NONLINEAR STOKES PHENOMENA ANALYTIC CLASSIFICATION INVARIANTS VIA GENERIC PERTURBATION

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Abstract. For a generic deformation of a two-dimensional holomorphic vector field with an elementary degenerate singular point (saddle-node) we express the Martinet - Ramis orbital analytic classification invariants of the nonperturbed field in terms of the limit transitions between the linearizing charts of singularities of the perturbed field. In the case, when the multiplicity of the singular point of the nonperturbed field is equal to 2, we show that the Martinet - Ramis invariants are limits of transition functions that compare appropriately normalized canonic first integrals of the perturbed vector field in the linearizing charts. We prove a generalization of this statement for higher multiplicity singularities. For a generic deformation of higher-dimensional holomorphic vector field with saddle-node singularity we show that appropriate sectorial central manifolds of the nonperturbed vector field are limits of appropriate separatrices of singularities of the perturbed field. We prove the analogues of the two first results for Ecalle-Voronin analytic classification invariants of one-dimensional conformal maps tangent to identity.

1. Introduction

Definition. Say that a singular point of a holomorphic vector field in a domain in $\mathbb{C}^2$ is *saddle-node*, if the correspondent linearization operator has exactly one zero eigenvalue.

Definition. Two holomorphic vector fields are said to be *orbitally analytically equivalent*, if there exists a biholomorphic diffeomorphism that maps phase curves of one vector field to those of the other.

Remark 1. Any germ of a holomorphic vector field in $(\mathbb{C}^2, 0)$ with saddle-node singularity at 0 is orbitally analytically equivalent to a vector field of the type

$$(1) \qquad \begin{cases} \dot{z} = z + O(|z|^2 + |t|^{k+1}) \\ \dot{t} = t^{k+1} \end{cases}$$

(see [1]).

One can ask the question: is it possible to transform a vector field (1) to a field that defines a differential equation with separable variables, more precisely, is is true

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that (1) is orbitally analytically equivalent to a direct sum of one-dimensional vector fields? Generically, the answer is "no" [1]. At the same time, there always exists a formal Taylor series $\tilde{z} = \widehat{H}(z,t)$ such that the correspondent formal change of the coordinates $(z,t) \mapsto (\tilde{z},t)$ is invertible and transforms (1) to a nonzero holomorphic function multiple of a unique vector field of the type

$$(1)_n \qquad \begin{cases} \dot{\tilde{z}} = \tilde{z} \\ \dot{t} = t^{k+1}(1+\lambda t^k)^{-1}, \end{cases}$$

$\lambda \in \mathbb{C}$, which is called the *formal normal form* of (1).

Generically, the normalizing series $\widehat{H}$ diverges [1]. At the same time, there exists a covering of a punctured neighborhood of zero in the t- plane by $2k$ sectors S_j, $j = 0, \dots 2k-1$, with the vertex at 0 having the following property: there exists a holomorphic change $\tilde{z} = H_j(z,t)$ of the variable z in a neighborhood of zero in the domain $\widetilde{S}_j = \mathbb{C} \times S_j$ that transforms phase curves of (1) to those of the formal normal form $(1)_n$ and has asymptotic Taylor series at 0 that coincides with $\widehat{H}$ ([1], [2]). Each sector S_j contains the radial ray with the argument $\frac{\pi(1+2j)}{2k}$ from the set $\{t^k \in i\mathbb{R}\}$ and its closure does not contain an additional ray from this set.

The nontriviality of transitions between these normalizing charts yields the obstruction for orbital analytic equivalence between (1) and $(1)_n$. This phenomena is called the nonlinear Stokes phenomena. The transitions present a complete system of orbital analytic equivalence invariants of saddle-node singularities of vector fields with a given formal normal form, which are called Martinet - Ramis invariants [1,2] and defined as follows. The formal normal form $(1)_n$ has the canonic first integral

$$I(\tilde{z},t) = \tilde{z} t^{-\lambda} e^{\frac{1}{kt^k}}.$$

The first integral I together with the sectorial normalizing coordinate changes H_j define canonic first integrals $I_j = I \circ H_j$ of (1) in the domains $\widetilde{S}_j$. (The sectors are numerated in the counterclockwise direction. We identify S_{2k} with S_0, and put also $I_{2k} = I \circ H_0$. In the definitions of all the integrals I_j we consider that for any pair of neighbor sectorial domains $\widetilde{S}_j$ and $\widetilde{S}_{j+1}$, $j < 2k$, the branch of the (multivalued) integral I in the latter is obtained from that in the former by counterclockwise analytic continuation.) Let ϕ_j be the transition functions that compare the first integrals I_j and I_{j+1} over the intersection component of the correspondent neighbor sectors S_j and S_{j+1}: $I_{j+1} = \phi_j \circ I_j$.

Remark 2. The functions $\phi_j(\tau)$ are holomorphic in $\mathbb{C}$ and have the type $\phi_j(\tau) = \tau + a_j$, if j is odd; for even j they are holomorphic in a neighborhood of zero and have unit derivative at 0: $\phi_j(\tau) = \tau + o(\tau)$, as $\tau \to 0$. Two germs of vector fields in $(\mathbb{C}^2, 0)$ of type (1) are orbitally analytically equivalent, iff they have the same formal normal form and the correspondent sets of $2k$ transition functions ϕ_j, $\tilde{\phi}_j$ $(j = 0, \dots, 2k-1)$ can be obtained from each other by simultaneous conjugation by multiplication by constant, $\phi_j(\tau) = c\tilde{\phi}_j(c^{-1}\tau)$, and cyclic permutation.

The equivalence classes of the transition function sets with respect to the last operations are called Martinet - Ramis invariants.

Example 1. Let $k = 1$. Then the above covering of a punctured neighborhood of zero consists of two sectors: S_0 and S_1 (Fig.1). The sector S_0 contains the

positive imaginary semiaxis, and its closure does not contain the negative imaginary semiaxis; the other sector S_1 possesses the same properties with the interchange of "negative" and "positive". In this case we have two transition functions ϕ_0 and ϕ_1 correspondent to the two components of the intersection of the sectors.

Fig.1

In the present paper we consider a generic continuous deformation

$$(1)_\varepsilon \qquad \begin{cases} \dot{z} = z(1+q(z,t,\varepsilon)) + g(t,\varepsilon)\prod_{i=0}^{k}(t-\alpha_i(\varepsilon)) \\ \dot{t} = \prod_{i=0}^{k}(t-\alpha_i(\varepsilon)) \end{cases}$$

of a field (1) in the class of holomorphic vector fields, $q(0,0,0)=0$. In particular, genericity means that the degenerate singularity 0 of the nonperturbed vector field (1) is split into $k+1$ nondegenerate linearizable singularities $(0,\alpha_i(\varepsilon))$ of the perturbed field, so that in a neighborhood of each of the latters there exists a local biholomorphic change of variables that transforms $(1)_\varepsilon$ to a linear field

$$(2) \qquad \begin{cases} \dot{z} = z \\ \dot{t} = \mu_i(\varepsilon)t, \end{cases}$$

where $\mu_i(\varepsilon)\to 0$, as $\varepsilon\to 0$. The field (2) has the canonic first integral $zt^{-\mu_i(\varepsilon)}$. The latter together with the linearizing change of variables defines a canonic first integral $I_{i,\varepsilon}$ of $(1)_\varepsilon$ in a neighborhood of the singular point $(0,\alpha_i(\varepsilon))$. The integral family $I_{i,\varepsilon}$ is uniquely defined up to multiplication by constant family.

The main result of the paper is the following. In the case, when $k=1$, for appropriately normalized canonic first integrals $I_{i,\varepsilon}$ we show that the transition functions that compare them in the intersection components of the correspondent linearization domains tend to the transition functions ϕ_i that define Martinet - Ramis invariants of the nonperturbed field (1), as $\varepsilon\to 0$. We prove a generalization of this statement for larger k. In fact, for any $k\in\mathbb{N}$ we prove, that appropriate single-valued branches of canonic first integrals of the perturbed field tend to appropriate sectorial canonic first integrals of the nonperturbed field. This result is stated in Subsection 2.3 and proved in Subsections 3.1-3.3 and 3.5.

In Section 4 we prove its higher-dimensional "central manifold" analogue.

A similar statement on the expression of Ecalle - Voronin moduli for analytic classification of germs of conformal maps $z\mapsto z+z^{k+1}+\dots$ via perturbation is stated in Subsection 2.4 and proved in Subsection 3.4. After obtaining a proof for some class of deformations, the author became aware that for deformations analytic in the parameter this statement was known to specialists (as A.Douady) and in the simplest case $k=1$ a proof was obtained by J.Martinet [4]. The proof in the general case was not published. After being acquainted with the known proofs, the author found that the method he used is quite different from that of the latters. It is similar to that of the previous statement on the Martinet-Ramis invariants.

Earlier a similar problem to obtain analytic classification invariants of linear differential equation with nonresonant irregular singularity in terms of its deformation was studied by R.Garnier [3], C.Zhang [5], J.-P.Ramis [12], A.Duval [13], the author [6] and others (see the references in [6]). In 1919 for a special class of such equations R. Garnier had obtained analytic classification invariants by studying some their particular deformations [3]. In 1984 V.I.Arnold conjectured that Stokes operators

of irregular singularity of linear differential equation can be expressed in terms of limit monodromy of its deformation. This conjecture was also independently stated by J.-P.Ramis in 1988 who proved that for classical confluenting family of hypergeometric equations Stokes operators of the nonperturbed equation are limits of transition operators between appropriately normalized monodromy eigenbases of the perturbed equation [12]. In 1991 A.Duval [13] proved the same statement for biconfluenting family of hypergeometric equations. In 1994 C.Zhang had obtained the expression of Garnier invariants via Stokes operators [5]. In fact, Garnier and Zhang implicitly proved the previous statement for the particular deformations of the equations considered by Garnier. In 1997 this statement was proved by the author for generic deformation of any irregular nonresonant singularity [6].

2. Main results

To state the main results, we recall some definitions, the Sectorial Normalization Theorem for saddle-node singularity of holomorphic vector field mentioned in Section 1, and the linearization Theorem for generic vector field with nondegenerate singularity.

2.1. Saddle-node vector fields. Sectorial Normalization Theorem.

Definition. Let $k \in \mathbb{N}$. An *imaginary dividing ray* is a radial ray from the set $\{t^k \in i\mathbb{R}\}$ in the complex plane with the coordinate t. A radial sector $S \subset \mathbb{C}$ is said to be *good*, if it contains a unique imaginary dividing ray and its closure does not contain an additional such ray. It is said to be j- good, if the correspondent ray has the argument $\frac{\pi(1+2j)}{2k}$ (the latter will be referred to, as r_j).

Example 2. Let $k = 1$. Then the imaginary dividing rays are the imaginary semiaxes. The sectors from Example 1 are good.

Remark 3. The imaginary dividing rays are exactly the rays, where the asymptotics of the first integral I of a vector field $(1)_n$ (Section 1), as $t \to 0$, $z = \text{const}$, is changed from zero to infinity.

Definition. Let (z, t) be a coordinate system in $\mathbb{C}^2$, U be a neighborhood of zero. For a subset S in the t- plane define $\widetilde{S} = (\mathbb{C} \times S) \cap U$.

Theorem 1. *(Sectorial Normalization Theorem* [1,2]*.) For any vector field (1) there exists a neighborhood U of zero in the phase space such that for any j- good sector S_j there exists a change $\tilde{z} = H_j(z, t)$ of the variable z holomorphic in the domain $\widetilde{S}_j$ and C^∞- smooth in its closure that transforms phase curves of (1) to those of its formal normal form $(1)_n$ (see Section 1) and has asymptotic Taylor series at 0 coinciding with the normalizing series $\widehat{H}$ from the beginning of Section 1.*

Definition. In the conditions of Theorem 1 the first integral of the field (1) in $\widetilde{S}_j$ obtained from the canonic first integral I of the formal normal form by sectorial normalizing chart is called the sectorial *canonic first integral*.

Remark 4. The sectorial canonic first integral is uniquely defined up to multiplication by constant [1].

2.2. Linearization Theorem.

Definition. The *characteristic number* of a holomorphic vector field in two-dimensional complex domain at its singular point is the ratio of the eigenvalues of the correspondent linearization operator.

Definition. A singular point of a holomorphic vector field in a two-dimensional complex domain is said to be *linearizable*, if the field is locally orbitally analytically equivalent to its linear part at the singularity in its neighborhood.

Theorem 2 [7]**.** *Let a holomorphic vector field in* $(\mathbb{C}^2, 0)$ *have singular point at 0 with finite nonreal characteristic number. Then the singular point is linearizable.*

Definition. The *canonic first integral* of a linear vector field

$$\begin{cases} \dot{z} = \lambda z \\ \dot{t} = \mu t, \end{cases}$$

with $|\lambda| > |\mu|$ is $zt^{-\frac{\lambda}{\mu}}$.

Definition. Let a holomorphic vector field have a linearizable singular point with linear part as in the previous Definition. The *canonic first integral* of the field at the singular point is the first integral obtained from the canonic first integral of the linear part by linearizing chart.

Remark 5. In the conditions of the previous Definition the canonic first integral is uniquely defined up to multiplication by constant.

2.3. Main results. Martinet - Ramis invariants are limits of transition functions between canonic integrals of generic perturbation.

We consider a continuous deformation $(1)_\varepsilon$ of a vector field (1) from Section 1 that in particular splits the degenerate singular point 0 into $k+1$ nondegenerate singularities $(0, \alpha_i(\varepsilon))$ of the perturbed field, i.e., $\alpha_i(\varepsilon) \neq \alpha_l(\varepsilon)$ for $i \neq l$, $\varepsilon \neq 0$. For a generic deformation $(1)_\varepsilon$ we prove the statement from the title of the Subsection.

Remark 6. We restrict ourselves by considering only deformations of the type $(1)_\varepsilon$ without loss of generality. Namely, in Subsection 3.7 of Section 3 we prove the following

Lemma 1. *Any continuous deformation of any vector field (1) in the class of holomorphic vector fields (without the condition that all the singularities of the perturbed field are nondegenerate, the only requirement is continuity) is orbitally analytically equivalent to a family of the type* $(1)_\varepsilon$ *via continuous family of changes of variables defined for all parameter values small enough.*

2.3.A. $k = 1$**.** Firstly let us state the main result in the case, when $k = 1$ (i.e., the perturbed field has two singularities). To do this, let us introduce the following

Definition. A family $(1)_\varepsilon$ as at the beginning of the Subsection is said to be *nondegenerate*, if the line in the t- plane passing through the singularity pair of the perturbed field intersects the real axis by angle bounded away from zero uniformly in ε small enough.

Remark 7. A family $(1)_\varepsilon$ is nondegenerate, iff the characteristic numbers of the singular points of the perturbed vector field have arguments bounded away from $\pi\mathbb{Z}$ uniformly in ε small enough. In particular, in this case the singularities of

the perturbed field are linearizable (Theorem 2). For small parameter values the singularities of the perturbed field satisfy the conditions of the Definition of the canonic first integral at a linearizable singular point from the previous Subsection. Thus, the canonic first integrals of the perturbed vector field at its singularities are well-defined for any ε small enough.

We consider nondegenerate families $(1)_\varepsilon$. Without loss of generality we consider that $\alpha_0 = -\alpha_1$.

Remark 8. Let a family $(1)_\varepsilon$ be as in the last item. Then for small parameter values $\operatorname{Im}\alpha_j(\varepsilon)$ have constant signs. The arguments $\arg\alpha_j$ are bounded away from $\pi\mathbb{Z}$. We consider that $\operatorname{Im}\alpha_0(\varepsilon) > 0$, $\operatorname{Im}\alpha_1(\varepsilon) < 0$.

To the family α_j we put into correspondence the sector $S_j \supset \{(-1)^j \operatorname{Im} t > 0\}$ from Example 1 (Fig.1, 2a), which contains $\alpha_j(\varepsilon)$ for all ε. We show that the branch of appropriately normalized (multivalued) canonic first integral of the perturbed field at its singular point $(0, \alpha_j(\varepsilon))$ converges to the sectorial canonic first integral of the nonperturbed vector field in $\widetilde{S}_j$.

Definition. Let $r > 0$, S be a radial sector in complex line with the coordinate t. Define $S^r = S \cap \{|t| < r\}$.

Theorem 3. *Let $k = 1$, $(1)_\varepsilon$ be a nondegenerate vector field family, $\alpha(\varepsilon)$, $-\alpha(\varepsilon)$ be the continuous families of the t- coordinates of its singular points, S be the sector correspondent to $\alpha(\varepsilon)$ from the item following Remark 8. There exist an $r > 0$, neighborhoods $U_z = \{|z| < 2\delta\}$, $U_t = \{|t| < \delta\}$ of zero in the z- and t- axes respectively and a family Ω_ε of simply connected subdomains of the disc U_t that contain $\alpha(\varepsilon)$, do not contain $-\alpha(\varepsilon)$ and possess the following properties:*

1)[1] the connected component of the intersection $\Omega_\varepsilon \cap (S^r \setminus [0, -\alpha(\varepsilon)])$ containing $\alpha(\varepsilon)$ tends to S^r, as $\varepsilon \to 0$.

2) Put $\Omega'_\varepsilon = \Omega_\varepsilon \setminus [-\alpha(\varepsilon), \alpha(\varepsilon)]$. The canonic first integral I_ε of the perturbed vector field $(1)_\varepsilon$ at its singularity $(0, \alpha(\varepsilon))$ is a multivalued holomorphic function in the domain $\widetilde{\Omega_\varepsilon} = U_z \times \Omega_\varepsilon$ with branching at the line $t = \alpha(\varepsilon)$. It is single-valued in its subdomain $\widetilde{\Omega'_\varepsilon}$. The restriction to the latter of appropriately normalized integral I_ε converges[2] to the canonic sectorial integral of the nonperturbed vector field in the domain $\widetilde{S^r} = U_z \times S^r$.

Theorem 3 is proved in Subsections 3.1-3.3 and 3.5.

Remark 9. The domain Ω_ε correspondent to $\alpha = \alpha_0$ is depicted at Fig.4a. It will be shown that it converges to the domain Ω depicted at Fig.4b bounded by a cardioid-like curve having inward cusp at 0 with two distinct tangent rays that bound a sector disjoint from both S and Ω.

Corollary 1. *In the conditions of Theorem 3 let α_j be singularity t- coordinate families of the vector fields $(1)_\varepsilon$, S_j^r, $\Omega'_\varepsilon = \Omega'_\varepsilon(j)$, be the correspondent sectors and domains from Theorem 3, C_0 be a connected component of the intersection $S_0^r \cap S_1^r$. Let ϕ be the correspondent Martinet-Ramis transition function between the sectorial*

[1] A family V_ε of planar domains is said to be convergent to a domain V, if both the maximal distance between a point of the boundary ∂V_ε and the whole boundary ∂V, and that with the interchange of V and V_ε tend to 0

[2] In the conditions of the previous footnote a family of functions holomorphic in V_ε depending continuously on the same parameter ε is said to be convergent (in V), if it converges uniformly in compact subsets of V

canonic integrals of the nonperturbed field from Theorem 1 over C_0. There exists a family C_ε of connected components of the intersections $S_0^r \cap S_1^r \cap \Omega'_\varepsilon(0) \cap \Omega'_\varepsilon(1)$ that converges to C_0 and possesses the following property. The transition functions between appropriately normalized canonic first integrals of the perturbed field in $\widetilde{C_\varepsilon}$ are well-defined and holomorphic in a domain (depending on ε) that converges to the definition domain of the function ϕ (either neighborhood of zero, or $\mathbb{C}$, see Remark 2). These transition functions converge to ϕ (footnote 2).

Fig.2a,b

2.3.B. $k \geq 2$**.** Now let us state the main result in the case, when $k \geq 2$. To do this, let us extend the Definition of nondegenerate deformation.

We consider families $(1)_\varepsilon$ with the following properties: the polynomial family $p(t,\varepsilon) = \prod_{i=0}^{k}(t - \alpha_i(\varepsilon))$ is differentiable in ε at $\varepsilon = 0$, $p'_\varepsilon(0,0) \neq 0$. Without loss of generality we consider that $\sum \alpha_j = 0$.

Remark 10. Let $p(t,\varepsilon)$ be a family as in the last item. Then its root polygon is asymptotically regular. This means that its radial homothethy image with the diameter 1 tends to a regular polygon (with the center at 0), as $\varepsilon \to 0$. The latter will be referred to, as Δ. Its vertex that is the limit of the homothethy images of the points $\alpha_j(\varepsilon)$ will be denoted by A_j. (We consider that α_j (and hence, A_j) are numerated in the counterclockwise direction.)

Definition. Let $k \in \mathbb{N}$. A *real (imaginary) dividing ray* or line in the complex plane with the coordinate t is a radial ray or line where $t^k \in \mathbb{R}$ (respectively, $t^k \in i\mathbb{R}$).

Definition. A family $(1)_\varepsilon$ as at the beginning of Subsection B is said to be *nondegenerate*, if no bissectrix of the correspondent limit regular polygon Δ from the previous Remark lies in a real dividing line.

Remark 11. For a family $(1)_\varepsilon$ as at the beginning of the Subsection, the nondegeneracy is equivalent to the characteristic number property of the singularities of the perturbed vector field from Remark 7. It is also equivalent to the condition that for any $i = 0, \dots, k$ there is a (unique) imaginary dividing ray r_{j_i} that has angle less than $\frac{\pi}{2k}$ with the radial ray of A_i (so, r_{j_i} is the imaginary dividing ray closest to the radial ray). For a nondegenerate family $(1)_\varepsilon$, to the singularity coordinate family α_i we put into correspondence a j_i- good sector S_{j_i} that contains r_{j_i} and the radial ray such that the latter has angles greater than $\frac{\pi}{2k}$ with the boundary rays of S_{j_i} (see Fig.2b for $k = 2$). Without loss of generality, everywhere below we consider that j_0 is equal to either 0, or 1 and the sequence $\{j_i\}_{i=0,\dots,k}$ is strictly increasing). One can achieve this by applying appropriate linear change of the variable t of the type $t \mapsto e^{2i\frac{\pi}{k}}t$. We put $\alpha_{k+1} = \alpha_0$, $j_{k+1} = j_0 + 2k$, $S_{j_{k+1}} = S_{j_0}$.

Remark 12. In the conditions of the previous Remark the sectors correspondent to the singularity families do not cover punctured neighborhood of zero: the total number of the imaginary dividing rays is $2k$, and that of the sectors is $k+1$ (each of the latters contains exactly one imaginary dividing ray). On the other hand, for each pair of sectors S_{j_i} and $S_{j_{i+1}}$, $i \leq k$, correspondent to neighbor singularity coordinate families $j_{i+1} - j_i \leq 2$, i.e., either the sectors intersect each other ($j_{i+1} - j_i = 1$), or they are intersected by the unique intermediate sector S_{j_i+1} of the covering ($j_{i+1} - j_i = 2$). Indeed, the angle between each neighbor pair of the A_i radial

rays from the previous Remark is equal to $\frac{2\pi}{k+1}$. Therefore, the angle between the correspondent imaginary dividing rays is less than $\frac{2\pi}{k+1}+\frac{\pi}{k}<3\frac{\pi}{k}$ (by definition), and hence, not greater than $2\frac{\pi}{k}$ (since this angle is a multiple of $\frac{\pi}{k}$ by definition). There are exactly two intersected sector pairs $(S_{j_i},\ S_{j_{i+1}})$ (i.e., with $j_{i+1}=j_i+1$).

Theorem 4. *Let $(1)_\varepsilon$ be a nondegenerate vector field family, $\alpha=\alpha_i$, $S=S_{j_i}$ be respectively its singularity coordinate family and the correspondent sector from Remark 11. There exist a neighborhood $U=U_z\times U_t$ of zero in the phase space and a family of domains $\Omega_\varepsilon=\Omega_\varepsilon(i)\subset U_t$ such that the triple $(\alpha,\Omega_\varepsilon,S)$ satisfies the statements of Theorem 3 with the change of $[0,-\alpha(\varepsilon)]$ in its statement 1) to $\bigcup_{\alpha_s\neq\alpha}[0,\alpha_s(\varepsilon)]$ and $[-\alpha(\varepsilon),\alpha(\varepsilon)]$ in its statement 2) to $\bigcup_s[0,\alpha_s(\varepsilon)]$.*

Theorem 4 is proved in Subsections 3.1-3.3 and 3.5.

Theorem 4 admits a Corollary analogous to Corollary 1 for the transition functions $\phi_{j_l,j_{l+1}}$ between the canonic first integrals of the nonperturbed field correspondent to the disjoint sector pairs S_{j_l}, $S_{j_{l+1}}$ (i.e., with $j_{l+1}=j_l+2$, the number of these pairs is equal to $k-1$, see the previous Remark). These transition functions are defined in the next item. All the Martinet-Ramis transition functions (different from those correspondent to the (two) intersected sector pairs $(S_{j_l},\ S_{j_{l+1}})$) are expressed in terms of the functions $\phi_{j_l,j_{l+1}}$ at the end of the next item.

Remark 13. Let (1) and $\bigcup_{j=0}^{2k-1}S_j$ be respectively a vector field and a covering from Section 1. Let S_l, S_{l+1} and S_{l+2} be a triple of consequent sectors from the covering, $l\leq 2k-1$ (we consider that $S_{2k}=S_0$, $S_{2k+1}=S_1$). Let I_l, I_{l+1}, I_{l+2} be the correspondent canonic integrals (defined consequently as in the item preceding Remark 2 in Section 1). Let ϕ_l and ϕ_{l+1} be the correspondent transition functions from Section 1. (Then $\phi_{2k}(\tau)=e^{-2\pi i\lambda}\phi_0(e^{2\pi i\lambda}\tau)$, by definition and since the analytic continuation by counterclockwise going around zero transforms the canonic integral I of the formal normal form to $e^{-2\pi i\lambda}I$.) In the case, when l is even, the zero level curves of the integrals I_l and I_{l+1} coincide over the intersection component of the correspondent sectors, i.e., continue analytically each other (Remark 2). In particular, I_l extends analytically counterclockwise to a neighborhood of this conjoint zero level curve in $\tilde{S}_{l+2}\cap\tilde{S}_{l+1}$, and the transition function $\phi_{l,l+2}(\tau)$ between the restrictions of I_l and I_{l+2} to this neighborhood ($I_{l+2}=\phi_{l,l+2}\circ I_l$) is well-defined and univalent in a neighborhood of zero. This transition function is equal to the composition $\phi_{l+1}\circ\phi_l$. *The transition functions ϕ_l and ϕ_{l+1} are expressed in terms of the composition function $\phi_{l,l+2}$* as follows:

$$\phi_{l+1}(\tau)=\tau+\phi_{l,l+2}(0),\qquad \phi_l(\tau)=\phi_{l,l+2}(\tau)-\phi_{l,l+2}(0).$$

Analogously, in the case, when l is odd, the inverse transition function $\phi_{l,l+2}^{-1}$ is well-defined and univalent in a neighborhood of zero and

$$\phi_{l,l+2}^{-1}=\phi_l^{-1}\circ\phi_{l+1}^{-1},\quad \phi_l(\tau)=\tau-\phi_{l,l+2}^{-1}(0),\quad \phi_{l+1}(\tau)=\phi_{l,l+2}(\tau+\phi_{l,l+2}^{-1}(0)).$$

Corollary 2. *In the conditions of Theorem 4 let α_l, α_{l+1} be neighbor singularity coordinate families of $(1)_\varepsilon$, $S_{j_l}^r$, $S_{j_{l+1}}^r$, $\Omega_\varepsilon(l)$, $\Omega_\varepsilon(l+1)$ be the correspondent sectors and domain families. Let $\Omega'_\varepsilon(s)=\Omega_\varepsilon(s)\setminus\bigcup_{l=0}^k[0,\alpha_l(\varepsilon)]$, $s=l,l+1$. In the case, when the sectors are intersected ($j_{l+1}-j_l=1$), the tuple $S_{j_l}^r$, $S_{j_{l+1}}^r$, $\Omega'_\varepsilon(l)$, $\Omega'_\varepsilon(l+1)$ satisfies the statements of Corollary 1. Otherwise, in the case, when the sectors are*

disjoint ($j_{l+1}-j_l=2$, see Remark 12) there exists a family C_ε of subdomains in the union $S^r_{j_l}\cup S^r_{j_l+1}$, $\alpha(\varepsilon)\in C_\varepsilon$, converging to the latter with the following properties:

1) Case, when j_l is even. The zero level curve of the canonic first integral $I_{l,\varepsilon}$ of the perturbed field at its singular point $(0,\alpha_l(\varepsilon))$ extends up to an analytic curve containing the graph $z=q_\varepsilon(t)$ of a function $q_\varepsilon(t)$ holomorphic in C_ε. The integral $I_{l,\varepsilon}$ extends analytically to a neighborhood of this graph. (This neighborhood intersects $\widetilde{\Omega'_\varepsilon}(l+1)$, which is the definition domain of the canonic integral $I_{l+1,\varepsilon}$, whenever ε is small enough.) Let $\phi_{l,\varepsilon}$, $I_{l+1,\varepsilon}=\phi_{l,\varepsilon}\circ I_{l,\varepsilon}$, be the transition function between the extended integral $I_{l,\varepsilon}$ and the integral $I_{l+1,\varepsilon}$ in the intersection of the correspondent definition domains. For appropriately normalized canonic integrals the function $\phi_{l,\varepsilon}$ is well-defined and holomorphic in a neighborhood of zero independent on ε (for all ε small enough) and tends to the composition $\phi_{j_l+1}\circ\phi_{j_l}$ of the Martinet-Ramis transition functions of the nonperturbed field (defined in Remark 2), as $\varepsilon\to0$.

2) Case, when j_l is odd. The same statements are valid with respect to the clockwise analytic extension of the integral $I_{j_{l+1},\varepsilon}$ and its zero level curve (from $\widetilde{\Omega'_\varepsilon}(l+1)$ to a subdomain of $\widetilde{\Omega'_\varepsilon}(l)$), the inverse transition function $\phi^{-1}_{l,\varepsilon}$ and the composition $\phi^{-1}_{j_l}\circ\phi^{-1}_{j_{l+1}}$ (the function ϕ_{2k} is defined in the previous Remark).

2.4. Analogues of main results for germs of conformal maps with identity linear part.

In this Subsection we state an analogue of Theorems 3 and 4 for germs of one-dimensional conformal maps in $(\mathbb{C},0)$ with the fixed point 0 of unit multiplier. To do this, let us introduce some definitions and recall the analogue of nonlinear Stokes phenomena for such germs.

Remark 14. Any germ as in the last item can be transformed to a one of the type $t\to t+2\pi i t^{k+1}(1+O(t))$, as $t\to0$, by linear change of the coordinate.

2.4.A. Nonlinear Stokes phenomena for one-dimensional conformal maps.

Definition. Two germs of conformal mappings $(\mathbb{C},0)\to(\mathbb{C},0)$ are said to be *analytically (formally) equivalent*, if there exists a germ of conformal diffeomorphism (an invertible formal Taylor series) that conjugates them.

Definition. Let f be a conformal map defined in a domain U, $D\subset U$ be its subdomain, v be a holomorphic vector field in U such that all its time s flow maps g^s_v with $0<s\leq1$ map D to U. The vector field v is said to be the *generator* of f, if its unit time flow map is f: $g^1_v|_D=f$.

The analytic classification of germs as in the previous Remark was obtained independently by J.Ecalle [8] and S.M.Voronin [9]. Namely, any such germ f has a formal generator, more precisely, is formally equivalent to the unit time flow map germ of a unique vector field of the type $v_\lambda(t)=2\pi i t^{k+1}(1+\lambda t^k)^{-1}$. This map is called the formal normal form of f. Generically the normalizing Taylor series diverges. At the same time there exists a covering of a punctured neighborhood of zero by $2k$ sectors S_j, $j=0,\dots,2k-1$, such that in each sector there exists a conformal diffeomorphism that conjugates f to its formal normal form with the following properties:

Theorem 5 [1,8,9]**.** *Let f be a germ as in Remark 14. Then for any j- good sector S_j (see the Definition in Subsection 2.1) there exists an $r > 0$ and a conformal map $h_j : S_j^r \to \mathbb{C}$ that conjugates f to its formal normal form with the following properties:*

1) the map h_j is C^∞ in the closure of the sector S_j^r, $h_j(0) = 0$, $h_j'(0) = 1$; its asymptotic Taylor series at 0 conjugates f to its formal normal form.

2) the maps h_j correspondent to different j may be chosen to have common asymptotic Taylor series at 0.

Definition. Let f, S_j^r, h_j be as in Theorem 5. The generator v_j of the restriction of the map f to S_j^r obtained from the correspondent vector field v_λ by the normalizing chart h_j is called the *canonic sectorial generator.*

Remark 15. The sectorial canonic generator of a germ f as in the previous Definition is the unique generator of f that is holomorphic in S_j^r and $o(t)$, as $t \to 0$. Let us prove the uniqueness of such a generator. By Theorem 5, it suffices to do this for the formal normal form, which will be now denoted by f. Let v be such a generator. Let us show that $v = v_\lambda$. For $r > 0$ small enough there exists a domain U_j containing the sector S_j^r invariant with respect to the field $-v_\lambda$ (in particular, f^{-1}-invariant) such that each backward f- orbit in U_j converges to 0 in the asymptotic direction of the imaginary dividing ray in S_j [1] (in particular, it fits S_j^r eventually). The space of these orbits is isomorphic to the double punctured Riemann sphere at 0 and ∞. The isomorphism is induced by the map $U_j \to \overline{\mathbb{C}} \setminus \{0, \infty\}$ defined by the formula $t \mapsto \tau = t^\lambda e^{-\frac{1}{kt^k}}$ (its right-hand side is equal to $e^{2\pi i T}$, where T is a complex time function of v_λ). Any generator of f in S_j^r that is $o(t)$, as $t \to 0$ (in particular, v), induces a holomorphic vector field in the whole orbit space (let us denote this field correspondent to v (v_λ) by v' (respectively, v'_λ)). A straightforward calculation shows that $v'(\tau) \to 0$, as τ tends to either 0, or ∞, so v' continues up to a holomorphic vector field in the whole Riemann sphere that vanishes at the punctured points, and so, it is linear. It has the form $\dot{\tau} = 2\pi i m \tau$, $m \in \mathbb{Z}$: the correspondent unit time flow map should be identity, since that of the generator v in S_j^r is f. The vector field v'_λ is $\dot{\tau} = 2\pi i \tau$, which follows from its definition. The two last statements imply that $v = m v_\lambda$, so, v is a generator of the m-th iteration of f. Therefore (since f is not periodic), $m = 1$ and $v = v_\lambda$.

Remark 16. In the conditions of the previous Definition let τ_j be the complex time function correspondent to the sectorial canonic generator in S_j normalized so that $\tau_j - \tau_{j+1} \to 0$, as $t \to 0$ in the connected component of the intersection of the neighbor sectors S_j and S_{j+1}, whenever $j \leq 2k-2$. Then $\tau_0 - \tau_{2k-1} \to -\lambda$, as $t \to 0$. (The possibility of such choice of time functions and the last statement are proved in [1].) We put $\tau_{2k} = \tau_0 + \lambda$. Consider the transition functions $\psi_j(\tau) = \tau_{j+1} \circ \tau_j^{-1}(\tau)$ between the time charts in the intersection components of the correspondent sectors, $j = 0, \dots, 2k-1$. The functions $\psi_j(\tau)$ extend up to functions holomorphic in appropriate half-planes $(-1)^j \operatorname{Im} \tau < c$, $c \in \mathbb{R}$, having the type

$$\psi_j(\tau) = \tau + \sum_{(-1)^j l < 0} c_j e^{2\pi i l \tau}.$$

Two germs f and $\tilde{f}$ as in Theorem 5 are analytically equivalent, iff the correspondent transition function sets ψ_j and $\widetilde{\psi_j}$ can be obtained one from the other by

simultaneous conjugation by addition of constant, $\widetilde{\psi_j}(\tau) = \psi_j(\tau - c) + c$, and subsequent cyclic permutation of the functions [1,8,9]. Thus, the equivalence classes of sets of transition functions with respect to the last operations are analytic classification invariants of the correspondent germs. They are called Ecalle - Voronin moduli [1,8,9].

2.4.B. Linearization of attracting and repelling fixed points. Canonical generators.

Theorem 6 [7]**.** *Let a one-dimensional conformal map have a fixed point with multiplier of nonunit module. Then it is conjugated to its linear part by conformal diffeomorphism in some neighborhood of the fixed point. The conjugating diffeomorphism is unique up to left composition with linear map.*

Definition. Let $\mu \in \mathbb{C} \setminus i\mathbb{R}_-$ be a number with a nonunit module. The canonic generator of the linear map $t \mapsto \mu t$ of the complex plane at its fixed point 0 is $\dot{t} = (\ln \mu)t$, where $\ln \mu$ is chosen so that $-\frac{\pi}{2} < \operatorname{Im} \ln \mu < \frac{3\pi}{2}$. The canonic generator of a nonlinear map at its fixed point with a multiplier as above is its generator obtained from the canonic generator of its linearization by the conjugating diffeomorphism from Theorem 6.

2.4.C. Ecalle - Voronin moduli via generators of perturbation.

We consider a continuous deformation

$$f_\varepsilon(t) = t + 2\pi i(1 + q(t, \varepsilon)) \prod_{i=0}^{k} (t - \alpha_i(\varepsilon))$$

of a conformal map f as in Remark 14, $q(0,0) = 0$. We show that for a generic deformation f_ε that in particular splits the degenerate fixed point of the nonperturbed map to $k+1$ nondegenerate linearizable fixed points $\alpha_i(\varepsilon)$ of the perturbed map appropriate sectorial canonic generators of the nonperturbed map are limits of canonic generators of the perturbed map.

Remark 17. Any continuous deformation of a map f as in Remark 14 in the class of conformal maps has the type f_ε.

Definition. A family f_ε as at the beginning of the Subsection is said to be *nondegenerate*, if the product $p(t, \varepsilon) = \prod_{i=0}^{k}(t - \alpha_i(\varepsilon))$ in its formula satisfies the condition of one of the two Definitions of nondegenerate family from Subsection 2.3 (one distinguishes the cases $k = 1$ and $k \geq 2$).

Remark 18. Let a family f_ε be nondegenerate. Then the multipliers $(f_\varepsilon)'(\alpha_j(\varepsilon))$ of the maps f_ε at their fixed points $\alpha_j(\varepsilon)$ tend to 1, as $\varepsilon \to 0$, and the differences $f'_\varepsilon(\alpha_j(\varepsilon)) - 1$ have arguments bounded away from $\frac{\pi}{2} + \pi\mathbb{Z}$ uniformly in all ε small enough. In particular, for all ε small enough the multipliers lie in the complement of unit circle.

Theorem 7. *Let f_ε be a nondegenerate family of conformal maps, $\alpha = \alpha_i(\varepsilon)$ be a continuous family of their fixed points, $S = S_{j_i}$ be a correspondent sector from either the item following Remark 8, if $k = 1$ (then $S_{j_i} = S_i$), or Remark 11 otherwise. There exist an $r > 0$ and a family of domains $\Omega_\varepsilon = \Omega_\varepsilon(i)$ depending on the same parameter containing $\alpha(\varepsilon)$ and no other fixed point with the following properties:*

1) The connected component of the intersection $\Omega_\varepsilon \cap (S^r \setminus \bigcup_{\alpha_s \neq \alpha}[0, \alpha_s(\varepsilon)])$ containing $\alpha(\varepsilon)$ converges to S^r, as $\varepsilon \to 0$.

2) The family v_ε of the canonic generators of the perturbed maps at $\alpha(\varepsilon)$ is well-defined for all parameter values small enough and each v_ε extends analytically to Ω_ε; the family v_ε converges to the sectorial canonic generator of the nonperturbed map in S^r, as $\varepsilon \to 0$.

Theorem 7 is proved in Subsection 3.4.

Corollary 3. *Let f_ε be a nondegenerate family of conformal maps, α_i be its continuous fixed point families, $S^r_{j_i}$, $\Omega_\varepsilon(i)$ be respectively the correspondent sectors and domains from Theorem 7. Let $\tau_{i,\varepsilon}(t)$ be the complex time functions correspondent to the canonic generators of f_ε at $\alpha_i(\varepsilon)$. The functions $\tau_{i,\varepsilon}$ are holomorphic and single-valued in the complements $\Omega'_\varepsilon(i) = \Omega_\varepsilon(i) \setminus \cup_{s=0}^k [0, \alpha_s(\varepsilon)]$. The families $\tau_{i,\varepsilon}$ of appropriately normalized time functions converge to the sectorial time functions τ_{j_i} from Remark 16 in $S^r_{j_i}$. Let α_i and α_{i+1} be neighbor fixed point families such that the correspondent sectors intersect each other, i.e., $S_{j_{i+1}} = S_{j_i+1}$ (we put $\tau_{2k+1} = \tau_1 + \lambda$, $\alpha_{k+1} = \alpha_0$, $\tau_{k+1,\varepsilon} = \tau_{0,\varepsilon} + \lambda$). There exists a family C_ε of connected components of the intersection $S^r_{j_i} \cap S^r_{j_{i+1}} \cap \Omega'_\varepsilon(i) \cap \Omega'_\varepsilon(i+1)$ that tends to the component of the intersection $S^r_{j_i} \cap S^r_{j_{i+1}}$, as $\varepsilon \to 0$. Let $\tau_{i+1,\varepsilon} \circ \tau^{-1}_{i,\varepsilon}$ be the transition function that compares the correspondent time functions in C_ε. The transition function is well-defined in a domain (depending on ε) that tends to a half-plane $(-1)^{j_i} \operatorname{Im} \tau < c$. The family of the transition functions converges to the correspondent Ecalle - Voronin transition function ψ_{j_i} of the nonperturbed map f_0 (see Remark 16, we put $\psi_{2k} = \psi_0$).*

A particular case of Theorem 7 for $k = 1$ and nondegenerate deformations f_ε analytic in ε such that $(f_\varepsilon)'_\varepsilon(0,0) \neq 0$ was proved in [4].

3. Proof of Theorems 3, 4, 7 and Lemma 1

3.1. Scheme of the proof of Theorems 3 and 4.

We prove Theorems 3 and 4 in Subsections 3.1-3.3 and 3.5. Another their proof that uses Theorem 7 and is more short is sketched in Subsection 3.6.

For the proof of Theorems 3 and 4 it suffices to show that there exists a family $\tilde{z} = H_\varepsilon(z,t)$ of changes of the variable z over neighborhoods Ω_ε of $\alpha(\varepsilon)$ satisfying statement 1) of Theorem 3 (4) that linearizes the differential equation in z of $(1)_\varepsilon$, i.e., transforms it to a family of equations of the type

$$\dot{\tilde{z}} = \tilde{z} g_\varepsilon(t),$$

and converges to the sectorial normalization $H(z,t)$ from Theorem 1 of the nonperturbed field. Then the canonic first integral of the perturbed field takes the form

$$\tilde{z} e^{-\int^t \frac{g_\varepsilon(\tau) d\tau}{\prod_j (\tau - \alpha_j(\varepsilon))}},$$

and by definition, its restriction to the correspondent domain $\widetilde{\Omega'_\varepsilon}$ from Theorem 3(4) will converge to the sectorial canonic first integral of the nonperturbed field.

Definition. In the conditions of Theorem 3(4) the *sectorial separatrix* Γ_0 of the nonperturbed field over the sector S is the zero level curve of the correspondent

sectorial canonic integral. The separatrix Γ_ε of the perturbed field is the zero level curve of its canonic integral at $(0, \alpha(\varepsilon))$.

Remark 19. Let (1) be a vector field as in Remark 1, U_z be a neighborhood of zero in the z- axis. There exists an $r > 0$ such that the sectorial separatrix Γ_0 of the nonperturbed field contains the graph $z = q(t)$, $t \in S^r$, of a U_z- valued function q holomorphic in the sector S^r and continuous in its closure, $q(0) = 0$. This is the unique phase curve of the nonperturbed field that contains such a graph. This follows from Theorem 1 and the fact that this is valid for the formal normal form.

Remark 20. Let $(1)_\varepsilon$ be a nondegenerate vector field family, $(0, \alpha(\varepsilon))$ be its continuous singularity family, U_z be a fixed neighborhood of zero in the z- line. For any $\varepsilon \neq 0$ the separatrix Γ_ε of the perturbed field at the singular point $(0, \alpha(\varepsilon))$ extends analytically to the latter and is tangent to the eigenline of the correspondent linearization operator transversal to the line $t = \alpha(\varepsilon)$ (the correspondent eigenvalue tends to 0, as $\varepsilon \to 0$). In particular, it contains the graph $z = q_\varepsilon(t)$ of a function q_ε holomorphic in a neighborhood of $\alpha(\varepsilon)$ (that depends on ε), $q_\varepsilon(\alpha(\varepsilon)) = 0$.

For the proof of the statement from the beginning of the Subsection on the existence and convergence of a linearization family H_ε we firstly show that the separatrices Γ_ε of the perturbed fields converge to the sectorial separatrix Γ_0 of the nonperturbed field:

Lemma 2. *Let $(1)_\varepsilon$, $\alpha(\varepsilon)$, S be as in Theorem 3 (4). There exist an $r > 0$ and neighborhoods $U_t = \{|t| < \delta\}$, $U_z = \{|z| < 2\delta\}$ of zero in the t- and z- lines respectively such that there exists a family Ω^1_ε of subdomains in U_t containing $\alpha(\varepsilon)$ and no other $\alpha_j(\varepsilon)$ satisfying statement 1) of Theorem 3 (4) with the following property: for all $\varepsilon \neq 0$ small enough the separatrix Γ_ε from the previous Definition contains the graph of a holomorphic function $z = q_\varepsilon(t)$ in $t \in \Omega^1_\varepsilon$ with values in U_z. The family q_ε converges to the function q from Remark 19 correspondent to the sectorial separatrix Γ_0 of the nonperturbed field.*

Lemma 2 is proved in Subsections 3.2 and 3.5. (A proof of its version is also implicitly contained in [6].) The correspondent domains Ω^1_ε are defined in the next Subsection (after Remark 21). In the case, when, $k = 1$, they are depicted at Fig.3a. They converge to a cardioid-like domain bounded by a Jordan curve with cusp at 0 depicted at Fig.3b.

In the proof of the statement from the beginning of the Subsection we use Lemma 2 and the following statement on the "uniqueness" of the sectorial z- variable linearizing chart for the nonperturbed field.

Proposition 1. *In the conditions of Theorem 1 let $U = \{|z| < 2\delta\} \times \{|t| < \delta\}$, $S = S_j$ be a good sector. Let Γ_0 be the zero level curve of the correspondent sectorial canonic integral. Let $H(z,t)$ be a holomorphic function in $\widetilde{S}$ univalent in z in the discs $t = const$ and vanishing in Γ_0 such that the change $\tilde{z} = H(z,t)$ of the variable z linearizes the differential equation in z of (1). Then H is obtained from the correspondent sectorial normalizing function H_j by multiplication by holomorphic function in t.*

Proof. It suffices to prove Proposition 1 for the formal normal form $(1)_n$ (Theorem 1), where the equation in z is already linear. Let H be as in Proposition 1. Let us show that H is linear in z. This will prove Proposition 1.

By definition, $H(0,t) \equiv 0$. Without loss of generality we consider that $\frac{\partial H}{\partial z}(0,t) \equiv 1$. One can achieve this by multiplying H by appropriate function in t.

We prove Proposition 1 by contradiction. Suppose the function H is not linear in z. Then it takes the form

$$H(z,t) = z(1 + z^l g(t) + \text{higher terms in } \ z).$$

The function g is bounded, which follows from univalency in z of the function H and distorsion theorem from [10]. On the other hand, it satisfies the equation

$$\frac{dg}{dt} = \frac{l}{t^{k+1}} g,$$

which follows from the conditions of Proposition 1 (the differential equation of the field in the new variable $\tilde{z} = H(z,t)$ is linear, as that in z). Therefore, g is a constant multiple of the function $e^{\frac{l}{kt^k}}$, which tends to infinity exponentially, as $t \to 0$ along a ray where the real part of its exponent is positive. The sector S, which is good, contains such a ray by definition: it contains an imaginary dividing ray, where this real part changes sign. This contradicts the boundedness of g. Proposition 1 is proved.

As it is shown in Subsections 3.3 nd 3.5 (Lemma 3 from the end of the present Subsection), there exist a disc U_z in the z- axis and z- variable linearizing charts $\tilde{z} = H_\varepsilon(z,t)$ for the perturbed fields (which vanish at Γ_ε) over appropriate domains Ω_ε^2 in the t- line (containing $\alpha(\varepsilon)$ and satisfying statement 1) of Theorem 3 (4)) univalent in $z \in U_z$ for any fixed $t \in \Omega_\varepsilon^2$. At the end of Subsection 3.3 we prove that the connected component Ω_ε of the intersection $\Omega_\varepsilon^1 \cap \Omega_\varepsilon^2$ (Ω_ε^1 is the domain from Lemma 2) that contains $\alpha(\varepsilon)$ satisfies statement 1) of Theorem 3 (4). This is the domain family we are looking for. One can choose such a family H_ε of linearizing charts to have unit derivatives in z at Γ_ε. Then the family $H_\varepsilon|_{\widetilde{\Omega_\varepsilon} \cap \widetilde{S}^r}$ is normal with respect to the compact convergence in $\widetilde{S^r}$ (Lemma 2 and normality of the space of the normalized univalent functions [10]). Using normality and Proposition 1, let us show that this family converges to the sectorial normalizing chart of the nonperturbed vector field over S^r (multiplied by a nonvanishing holomorphic function in t). This will prove Theorems 3 and 4.

By construction, the limit of each convergent sequence H_{ε_n}, $\varepsilon_n \to 0$, is a z-linearizing chart over S^r for the nonperturbed field that satisfies the conditions of Proposition 1 and has unit derivative in z at Γ_0. By Proposition 1, this chart is obtained from the canonic normalizing chart by multiplication by nonvanishing holomorphic function in t. The chart satisfying conditions of Proposition 1 and having unit derivative in z at Γ_0 is unique, which follows immediately from Proposition 1. In particular, all the limits of convergent sequences coincide, and hence, the family H_ε converges to a chart as at the end of the previous item. This proves Theorems 3 and 4 modulo Lemma 2, the following Lemma and statement 1) of Theorem 3 (4) for the correspondent domains Ω_ε from the previous item. The last statement is proved at the end of Subsection 3.3.

Lemma 3. *In the conditions of Lemma 2 there exist an $r > 0$, neighborhoods U_z, U_t of zero in the z- and t- axes respectively, a family of subdomains $\Omega_\varepsilon^2 \subset U_t$ satisfying statement 1) of Theorem 3 (4) such that there exists a family $H_\varepsilon(z,t)$*

of functions holomorphic in $\widetilde{\Omega^2_\varepsilon} = U_z \times \Omega^2_\varepsilon$ and univalent in $z \in U_z$ in the discs $t = const$ such that the coordinate change $\tilde{z} = H_\varepsilon(z,t)$ linearizes the differential equation in z of the perturbed field.

Lemma 3 is proved in Subsections 3.3 and 3.5.

3.2. Continuity of the separatrices. Proof of Lemma 2.

In the proof of Lemma 2 we use the uniqueness statement of Remark 19 from the previous Subsection.

Let q and q_ε be the functions from Remarks 19 and 20 correspondent respectively to the sectorial separatrix of the nonperturbed vector field and the separatrix of the perturbed field. For the proof of convergence $q_\varepsilon \to q$ in appropriate sector S^r we show that the function family q_ε is equicontinuous in a family of domains convergent to S^r. Then they form a normal family (Arzela-Ascoli theorem). Each limit of sequence $\{q_{\varepsilon_n}\}$, $\varepsilon_n \to 0$, compactly convergent in S^r is holomorphic in S^r and extends continuously to its closure. The graph of each limit is contained in a phase curve of the limit (nonperturbed) field. Therefore, by the uniqueness statement of Remark 19, these limits coincide with the function q correspondent to the sectorial separatrix Γ_0. This will prove Lemma 2.

The equicontinuity statement from the last item (and hence, Lemma 2 also) is implied by the following

Lemma 4. *In the conditions of Lemma 2 there exist an $r > 0$, a neighborhood U_z of zero in the z- line and a family Ω^1_ε of domains in the t- line containing $\alpha(\varepsilon)$ and satisfying statement 1) of Theorem 3 (4) such that the separatrix Γ_ε of the perturbed vector field at $(0, \alpha(\varepsilon))$ contains the graph $z = q_\varepsilon(t)$ of a function $q_\varepsilon(t)$, $q_\varepsilon(\alpha(\varepsilon)) = 0$, holomorphic in Ω^1_ε with values in U_z that satisfies the following inequalities:*

$$(3) \qquad |\frac{dq_\varepsilon}{dt}(t)| < 1, \quad |q_\varepsilon(t)||_{\alpha(\varepsilon)\neq t \in \Omega_\varepsilon} < |t - \alpha(\varepsilon)|.$$

Proof. Before the proof of Lemma 4, let us introduce the following

Definition. Let $(1)_\varepsilon$ be a nondegenerate family, $\theta \in \mathbb{R}$. Define $v_\theta(\varepsilon)$ to be the vector field family that is the $e^{i\theta}$-th multiple of $(1)_\varepsilon$. Define $w_\theta(\varepsilon)$ to be the projection image in the t- line of the restriction $v_\theta(\varepsilon)|_{z=0}$.

The inequalities of Lemma 4 are equivalent to the statements that the tangent lines to Γ_ε are contained in the tangent cone field $K = (|\dot{t}| > |\dot{z}|)$ and Γ_ε is contained in the cone $\widetilde{K} = \{|t - \alpha(\varepsilon)| > |z|\}$:

$$(4) \qquad T\Gamma_\varepsilon \subset K, \qquad \Gamma_\varepsilon \subset \widetilde{K}.$$

Inclusions (4) hold a priori in a neighborhood of the singular point $(0, \alpha(\varepsilon))$ (that depends on ε). To show that they hold in a large domain, we consider a vector field family $v_\theta(\varepsilon)$ from the previous Definition with the following properties:

1) the eigenvalue of the linearization operator of the field $v_\theta(\varepsilon)$ at the eigenline tangent to the line $t = \alpha(\varepsilon)$ has negative real part bounded away from 0 uniformly in ε small enough;

2) the other eigenvalue has positive real part, and its argument is bounded away from $\frac{\pi}{2} + \pi\mathbb{Z}$ uniformly in ε small enough.

Remark 21. Let $(1)_\varepsilon$ be a nondegenerate family, $\alpha(\varepsilon)$ be its continuous singularity coordinate family. There exists a $\theta \in \mathbb{R}$, $\frac{\pi}{2} < \theta < \frac{3\pi}{2}$, that satisfies the conditions 1) and 2) from the two previous items. Indeed, for θ satisfying the last inequality condition 1) holds by definition. Condition 2) is satisfied, e.g., for $\theta_0 = \pi \pm \frac{\pi}{2}$ with appropriately chosen sign (dependently on the choice of the family $\alpha(\varepsilon)$), which follows from nondegeneracy of $(1)_\varepsilon$. Thus, 2) holds also for all θ close enough to θ_0, which may be chosen to satisfy the previous inequality. Under the conditions 1) and 2) $(0, \alpha(\varepsilon))$ is a hyperbolic singularity of the field $v_\theta(\varepsilon)$, with 1-dimensional stable and unstable manifolds: the former is contained in the line $t = \alpha(\varepsilon)$; the latter is contained in Γ_ε. The singular point $\alpha(\varepsilon)$ of the field $w_\theta(\varepsilon)$ is repelling.

We use the fact that under conditions 1) and 2) there exists a neighborhood U of zero in the phase space such that for all ε small enough the cone field K (and hence, $\widetilde{K}$ as well) is $v_\theta(\varepsilon)$- invariant in U (proposition 2 in [6]). This together with the local inclusions (4) in a neighborhood of the singular point $(0, \alpha(\varepsilon))$ implies that the latters hold in the trajectories of $v_\theta(\varepsilon)$ in $\Gamma_\varepsilon \cap U$ that go from $(0, \alpha(\varepsilon))$ (denote by Ω''_ε the subdomain of the separatrix Γ_ε saturated by these trajectories). Without loss of generality we consider that $U = U_z \times U_t$, $U_z = \{|z| < 2\delta\}$, $U_t = \{|t| < \delta\}$. Then inclusions (4) hold over the following domains.

Definition of the domains Ω^1_ε. In the conditions of Theorem 3(4) and the Definition from the beginning of the proof of Lemma 4 let θ satisfy the condition 2) preceding Remark 21. Let U_t be a fixed disc centered at zero in the t- line. Define $\Omega^1_\varepsilon \subset U_t$ to be the subdomain saturated by the trajectories of the field $w_\theta(\varepsilon)$ in U_t that go from $\alpha(\varepsilon)$. (In the case, when $k = 1$, the phase portrait of the field $w_\theta(\varepsilon)$ and the domain Ω^1_ε are depicted at Fig.3a. The boundary of the domain Ω^1_ε consists of the three following parts: a) the arc of the boundary ∂U_t where the field $w_\theta(\varepsilon)$ is directed outside U_t bounded by the pair of tangency points of the field with ∂U_t; b) the pair of semitrajectories of the field that start at these points and tend to its other singularity.)

Inclusions (4) hold in Ω''_ε (see the item preceding the Definition). The latter is 1-to-1 projected onto Ω^1_ε. This follows from the inclusion $\Omega''_\varepsilon \subset \widetilde{K}$ and the definition of the neighborhood U (exactly in the same way, as in the proof of lemma 1 in [6]). Therefore, Ω''_ε is the graph of a function $q_\varepsilon(t)$ holomorphic in Ω^1_ε and satisfying inequalities (3).

Now for the proof of Lemma 4 it suffices to show that one can choose θ satisfying conditions 1) and 2) so that the correspondent family Ω^1_ε satisfies statement 1) of Theorem 3 (4). To do this, we prove that appropriate family of subdomains of Ω^1_ε converges to a domain Ω^1 bounded by a Jordan curve having cusp at 0 of the following type. In the case, when $k = 1$, the branches of the cusp are tangent to the ray $-R(\theta)$ with the argument $\pi - \theta$ disjoint from Ω^1 (Fig.3b). We show that one can choose θ satisfying conditions 1) and 2) so that the correspondent domain Ω contains the sector S^r for appropriate $r > 0$. To do this, we prove the possibility to choose θ as above so that the ray $-R(\theta)$ punctured at 0 lies outside the closure of the sector S. In the case, when, $k \geq 2$, the cusp branches of $\partial\Omega^1$ are tangent to a pair of rays that bound a sector disjoint from Ω^1. We show that one can choose θ satisfying conditions 1) and 2) so that the closure punctured at 0 of the last sector

is disjoint from that of S. Then Ω will contain S^r for appropriate $r > 0$.

Definition. Let $k \in \mathbb{N}$, $k \geq 1$, $\theta \in \mathbb{R}$. Define $R_m(\theta)$ to be the radial ray with the argument $\frac{-\theta+\pi m}{k}$.

In the proof of the statements from the item preceding the previous Definition we use the following properties of the nonperturbed vector field $w_\theta(0)$.

Remark 22. In the conditions of the previous Definition each trajectory of the vector field $\dot{t} = e^{i\theta}t^{k+1}$ has one of the three following types: (i) a ray $R_m(\theta)$ with m even oriented from 0 to infinity (repelling ray); (ii) a ray $R_m(\theta)$ with m odd oriented towards 0 (attracting ray); (iii) a Jordan curve that goes from zero with asymptotic tangency to a ray from (i) and returns to 0 with the asymptotic tangency to a neighbor ray from (ii).

Corollary 4. *In the conditions of the previous Remark let R_m be a repelling ray, U_t be a disc centered at 0. Then the subdomain $\Omega_m \subset U_t$ saturated by the trajectories of the field in U_t that go from 0 with asymptotic tangency to the ray R_m is bounded by a Jordan curve that has a cusp at 0 with the two tangency rays $R_{m\pm1}$ disjoint from Ω_m. (This subdomain is called the repelling domain of R_m. In the case, when $k = 1$, it is depicted at Fig.3b.) Its boundary consists of the three following parts: an arc of ∂U_t where the field is directed outside U_t bounded by tangency points of the field with ∂U_t; two semitrajectories that start at these points and converge to 0 with tangency to the rays $R_{m\pm1}$.*

Example 3. Let $k = 1$. Then in the conditions of the previous Remark the only repelling ray $R(\theta)$ is that with the argument $-\theta$, and the opposite one $(-R(\theta))$ is the only attracting ray. The correspondent repelling domain Ω_m from Corollary 4 is bounded by a cardioid-like Jordan curve having inward cusp at 0 with tangency to the attracting ray, which is disjoint from Ω_m (Fig.3b).

Fig.3 a,b

For appropriate repelling domain Ω^1 from Corollary 4 correspondent to the nonperturbed field $w_\theta(0)$ we show that the domains $\Omega^1_\varepsilon \cap \Omega^1$ converge to Ω^1. This is implied by the following

Proposition 2. *Let $\theta \in \mathbb{R}$. Let $w(\varepsilon)$: $\dot{t} = e^{i\theta}(t - \alpha_0(\varepsilon)) \dots (t - \alpha_k(\varepsilon))$ be a continuous family of vector fields in complex line with the coordinate t having $k+1$ distinct singularities $\alpha_i(\varepsilon)$ that tend to 0, as $\varepsilon \to 0$. In the case, when $k \geq 2$, let the singularity polygon satisfy the asymptotic regularity statement of Remark 10. Let U_t be a neighborhood of zero in the phase line. Let $\alpha(\varepsilon) = \alpha_j(\varepsilon)$ be a continuous singularity family. Let the multiplier of the field $w(\varepsilon)$ at $\alpha(\varepsilon)$ have positive real part and its argument be bounded away from $\frac{\pi}{2} + \pi\mathbb{Z}$. Let $\Omega_\varepsilon \subset U_t$ be the correspondent family of subdomains from the Definition following Remark 21. Then there exists a repelling ray R (see the previous Remark) correspondent to the nonperturbed field $w(0)$ such that for all ε small enough R is the closest of all the repelling and attracting rays to the radial ray of $\alpha(\varepsilon)$. Let Ω be the correspondent repelling domain from Corollary 4. In the case, when $k = 1$, the family Ω_ε tends to Ω, as $\varepsilon \to 0$. In the case, when $k \geq 2$, so does the family of the connected components of the intersections $\Omega_\varepsilon \cap \Omega$ containing $\alpha(\varepsilon)$. In both cases there exists an $r' > 0$ such that each Ω_ε contains the segment of the radial ray of the point $\alpha(\varepsilon)$ joining the latter to the point of the circle $|t| = r'$.*

A proof of Proposition 2 except for that of its two last statements is implicitly contained in subsections 5.A and 6 of the paper [6]. Its last statements are proved in Subsection 3.5.

For simplicity, we consider only the case, when $k = 1$, and prove the existence of θ such that Ω^1_ε satisfy statement 1) of Theorem 3. The proof remains valid for larger k with obvious changes.

We show that one can choose θ satisfying conditions 1) and 2) from the item preceding Remark 21 so that the attracting ray $-R(\theta)$ from the previous Example punctured at 0 lies outside the closure of the sector S correspondent to $\alpha(\varepsilon)$. Then the correspondent "limit repelling domain" from Proposition 2 (it will be referred to, as Ω^1) will contain a sector S^r for some $r > 0$. Thus, a connected component of the intersection from statement 1) of Theorem 3 tends to S^r. This component contains $\alpha(\varepsilon)$. Indeed, for all ε small enough it contains the segment from the last statement of Proposition 2 correspondent to $r' < r$. This follows from the fact that the segment end different from $\alpha(\varepsilon)$ is contained in the sector S^r and bounded away from its boundary (by definition) uniformly in all small ε. This proves statement 1) of Theorem 3 modulo the two last statements of Proposition 2.

It suffices to prove the first statement from the last item for the argument θ_0 from Remark 21 instead of θ, since θ satisfying conditions 1) and 2) may be chosen arbitrarily close to θ_0 (Remark 21). The statement for θ_0 follows from the fact that $R(\theta_0)$ is the imaginary dividing ray that lies in S (so, the opposite imaginary dividing ray $-R(\theta_0)$ lies outside the sector S, since the latter is good). This follows from the definitions of θ_0 (Remark 21) and the correspondent ray $R(\theta_0)$ (Example 3). Statement 1) of Theorem 3 together with Lemma 2 is proved modulo the two last statements of Proposition 2.

3.3. Proof of Lemma 3. The end of the proof of Theorems 3 and 4 modulo Proposition 2.

Firstly let us prove Lemma 3. In its proof we use the following

Remark 23. Let

$$\begin{cases} \dot{z} = z + O(|z|^2 + |t|^2) \\ \dot{t} = \lambda t(1 + g(t)), \end{cases}$$

be a holomorphic vector field in a neighborhood of zero in $\mathbb{C}^2$, $g(0) = 0$, $\lambda \notin \mathbb{R}$. There exists a unique change $\tilde{z} = H(z,t)$ of the variable z in a neighborhood of zero that linearizes in z the correspondent (first) differential equation up to multiplication by nonzero function in t.

A z- linearizing chart $\tilde{z} = H_\varepsilon(z,t)$ for the perturbed vector field exists a priori in a neighborhood $U'_z \times U'_t$ of the singular point $(0, \alpha(\varepsilon))$ (that depends on ε). Without loss of generality we consider that it is univalent in z in the discs $U'_z \times t$ and has unit derivative in z at Γ_ε. To show that it continues to a large domain, we consider a vector field $v_\theta(\varepsilon)$ (correspondent to another value θ) that satisfies the conditions 1) and 2) preceding Remark 21 from the previous Subsection with the change of "negative real part" in the condition 1) to "positive real part". Then its singular point $(0, \alpha(\varepsilon))$ is repelling, i.e., it is asymptotically stable with respect to the opposite field $-v_\theta(\varepsilon)$. Denote by g^τ the time τ flow map of the field $v_\theta(\varepsilon)$. We use the fact that for any τ and linearizing function H_ε (defined in a neighborhood of the singularity) the composition $H_{\varepsilon,\tau} = H_\varepsilon \circ g^\tau$ also defines a change of the coordinate z linearizing the correspondent differential equation. Thus, it is obtained

from H_ε by multiplication by function in t (the previous Remark). Therefore, by definition, one has $H_\varepsilon = \frac{H_{\varepsilon,\tau}}{((H_{\varepsilon,\tau})'_z)|_{\Gamma_\varepsilon}}$. Let $U = U_z \times U_t$ be a fixed bidisc centered at zero in the phase space. The last formula defines the analytic continuation of H_ε along the trajectories of $v_\theta(\varepsilon)$ in U_t that go from $(0, \alpha(\varepsilon))$, thus, to the domain saturated by the latters. This domain will be referred to, as Ω''_ε. For appropriate neighborhoods U_z and U_t and all ε small enough Ω''_ε is the Cartesian product $\widetilde{\Omega^2_\varepsilon}$ of the disc U_z and the domain from the previous Definition (it will be referred to, as Ω^2_ε) correspondent to the new θ: namely, if the field $-v_\theta(\varepsilon)$ is directed inside the domain $\widetilde{\Omega^2_\varepsilon}$. This is the case, if the field $-v(\theta)$ is directed inside the cylinder $U_z \times \mathbb{C}$ over U_t (by definition). The possibility of choice of neighborhoods U_z and U_t satisfying the last condition for all ε small enough follows immediately from the choice of θ (the modified condition 1)) and the continuity of the family $(1)_\varepsilon$. Then by construction, the analytic continuation of the function H_ε to $\widetilde{\Omega^2_\varepsilon}$ is linearizing and univalent in z. This follows from definition and the fact that the fibration by parallel lines $t =$ const is preserved under the flow g^τ. This proves Lemma 3 for the domains Ω^2_ε modulo statement 1) of Theorem 3 (4).

Similarly, as in the proof of Lemma 2, one can choose θ as in the previous Definition so that the family Ω^2_ε satisfies statement 1) of Theorem 3 (4), more precisely, the correspondent repelling domain from Proposition 2 contains the sector S^r for appropriate $r > 0$. This proves Lemma 3 (modulo the last statements of Proposition 2).

Now Theorems 3 and 4 are proved modulo statement 1) of Theorem 3 (4) for the following domains Ω_ε (see the discussion in Subsection 3.1 preceding the statement of Lemma 3).

Definition of the domains Ω_ε. In the conditions of Theorem 3(4) and the Definition following Lemma 4 let $\theta_1, \theta_2 \in \mathbb{R}$ satisfy condition 2) preceding Remark 21, θ_1 satisfies condition 1) from the same place, θ_2 satisfies its modification with the change of "positive real part" to "negative real part". Let U_t be a disc centered at 0 in the t- axis. Let Ω^1_ε, Ω^2_ε be the domains from the Definition of the domains Ω^1_ε in the previous Subsection correspondent to θ_1 and θ_2 respectively. Define Ω_ε to be the connected component of the intersection $\Omega^1_\varepsilon \cap \Omega^2_\varepsilon$ that contains $\alpha(\varepsilon)$. (In the case, when $k = 1$, the boundary of the domain Ω_ε consists of the three following parts: an arc of ∂U_t where both fields $w_{\theta_1}(\varepsilon)$ and $w_{\theta_2}(\varepsilon)$ are directed outside U_t bounded by the tangency points $F_1(\varepsilon)$ and $F_2(\varepsilon)$ respectively of these fields with ∂U_t; two arcs of their semitrajectories (tangent to ∂U_t) that start at $F_1(\varepsilon)$ and $F_2(\varepsilon)$ respectively and have a common end, see Fig.4a.)

Fig. 4 a,b

For appropriate values θ_1 and θ_2 from the previous Definition the correspondent family Ω_ε satisfies statement 1) of Theorem 3 (4). This is the case, provided that so do Ω^1_ε and Ω^2_ε (more precisely, the correspondent limit domains Ω^1 and Ω^2 from Proposition 2 contain the sector S^r for appropriate $r > 0$). Indeed let Ω be the connected component of the intersection $\Omega^1 \cap \Omega^2$ that contains the sector S^r. The domain Ω is bounded by a Jordan curve that consists of the three following parts: an arc of ∂U_t where both $w_{\theta_1}(0)$ and $w_{\theta_2}(0)$ are directed outside U_t bounded by the tangency points F_1 and F_2 respectively of the latters with ∂U; a pair of positive semitrajectories L_1 and L_2 of these fields that start at the points F_1 and F_2 respectively and converge to 0 (Fig.4b). In the case, when $k = 1$, $\Omega_\varepsilon \to \Omega$,

as $\varepsilon \to 0$, which was implicitly proved in subsection 5.B of [6]. In the case, when $k \geq 2$, the connected component of the intersection $\Omega_\varepsilon \cap \Omega$ containing $\alpha(\varepsilon)$ tends to Ω, as $\varepsilon \to 0$. Indeed, by the convergence statement of Proposition 2 applied to $\Omega^i_\varepsilon \cap \Omega^i$, $i = 1, 2$, some connected component of the intersection $\Omega_\varepsilon \cap \Omega$ converges to Ω. This together with the last statement of Proposition 2 implies that this component contains $\alpha(\varepsilon)$, whenever ε small enough, as in the discussion preceding the last item of Subsection 3.2. Analogously statement 1) of Theorem 3 (4) follows from the convergence statements for the domains Ω_ε and the last statement of Proposition 2 for the domains Ω_ε (it holds for both Ω^i_ε). Theorems 3 and 4 are proved modulo the two last statements of Proposition 2.

3.4. Proof of Theorem 7.

The method of the proof of Theorem 7 is similar to that of Theorems 3 and 4.

Without loss of generality we consider that in the conditions of Theorem 7 $\alpha(\varepsilon)$ is a repelling fixed point family for f_ε. One can achieve this by changing f_ε to their inverse maps and subsequent linear coordinate change. The generator v_ε of the perturbed map f_ε is a priori defined in a neighborhood of $\alpha(\varepsilon)$ (that depends on ε) and is f_ε- invariant. It extends analytically to the domain Ω_ε from the next Definition. The continuation is made along the orbits of f_ε that go from $\alpha(\varepsilon)$ by applying the iterations of f_ε to the generator defined near $\alpha(\varepsilon)$.

Definition of the domains Ω_ε. Let f_ε and $\alpha(\varepsilon)$ be as in the last item and Theorem 7. Let $U = \{|t| < \delta\}$ be a fixed neighborhood of zero where f_ε are holomorphic and have the only fixed points $\alpha_j(\varepsilon)$ for all ε. For any ε define $\Omega_\varepsilon \subset U$ to be the subdomain saturated by the orbits of the map f_ε in U that go from $\alpha(\varepsilon)$ (see the reversed Fig.3a for $k = 1$).

We show that the domains Ω_ε satisfy the statements of Theorem 7.

Without loss of generality, we consider that the argument family of the fixed points $\alpha(\varepsilon)$ has a limit, as $\varepsilon \to 0$ (in the case, when $k \geq 2$, this follows from the definition of nondegeneracy). Each parameter value sequence $\varepsilon_n \to 0$ contains a subsequence where the arguments $\arg \alpha(\varepsilon)$ converge. Therefore, the convergence of the generators v_ε of the perturbed maps in these subsequences will imply their convergence in the whole parameter space.

By $p(t, \varepsilon)$ denote the vector field family $\dot{t} = \prod_{i=0}^k (t - \alpha_i(\varepsilon))$.

Firstly we prove statements of Theorem 7 for a smaller good sector $C \subset S$. For the proof of convergence of the generators v_ε we show that there exist $r, r', b > 0$, a family of f_ε^{-1}- invariant subdomains $D_\varepsilon \subset \Omega_\varepsilon$ containing $\alpha(\varepsilon)$, and a good sector $C \subset S$ with the following properties:

1) the family D_ε converges to a (connected) domain D_0 containing the sector C^r;
2) the inequality $|v_\varepsilon(t)| \leq b|p(t, \varepsilon)|$ holds in D_ε for all ε small enough;
3) for any ε small enough the domain D_ε contains the segment of the radial ray of the point $\alpha(\varepsilon)$ joining the latter to the point of the circle $|t| = r'$.

(In the case, when $k = 1$, for $\varepsilon \neq 0$ the domain D_ε will be a disc symmetric with respect to the line passing through the fixed point pair of the map f_ε, the limit domain D_0 will be the maximal disc in U symmetric with respect to this line whose boundary contains 0 (Fig. 5a,b).) Then the family v_ε converges to the sectorial generator v_0 of the nonperturbed map in D_0 (in particular, in C^r). This is proved by using the bound 2), normality of the family v_ε (implied by 2)) and uniqueness of the sectorial generator (Remark 15), analogously to the proof of Lemma 2 (the discussion preceding Lemma 4 in Subsection 3.2).

Now let us show how the convergence statement in the smaller good sector C together with the previous statement 3) implies Theorem 7 (for the larger sector S). Let r_j be the imaginary dividing ray contained in the sector S, which has the argument $\frac{\pi}{2k}+\frac{\pi j}{k}$. Consider the subdomain $\Omega\subset U$ saturated by the orbits of the limit map f_0 in U that go from zero in the asymptotic direction of the ray r_j. The domain $\Omega\subset U$ is bounded by Jordan curve that has cusp at 0 with tangency to the two imaginary dividing rays neighbor to r_j. This is implicitly proved in [1]. (In the case, when $k=1$, these two rays coincide, see the reversed Fig.3b.) In particular, $\Omega\supset S^r$ for appropriate $r>0$. A connected component of the intersection $\Omega_\varepsilon\cap\Omega$ converges to Ω, as $\varepsilon\to 0$. This follows from the definition of the domains Ω_ε, the property 1) of the domains D_ε and the fact that Ω is saturated by the f_0- orbits that start in C^r (by definition). (In the case, when $k=1$, $\Omega_\varepsilon\to\Omega$. The proof of this statement is omitted to save the space.) Therefore, the same convergence statement holds with the change of $\Omega_\varepsilon\cap\Omega$ to the intersection from statement 1) of Theorem 7. The sectorial generator v_0 of the nonperturbed map extends analytically from C^r to Ω by applying iterations of the map f_0 to its restriction to C^r. The analogous continuation of the generator v_ε to Ω_ε along the f_ε- orbits starting in $D_\varepsilon\cap C^r$ converges to v_0 in Ω. This follows from continuity of the family f_ε and the convergence of v_ε in C^r. This proves Theorem 7 for the sector S^r modulo the statement that the convergent component of the correspondent intersection contains $\alpha(\varepsilon)$. This follows from the convergence of the component under consideration and the previous property 3) of D_ε (analogously to the discussion preceding the last item of Subsection 3.2).

In the proof of the previous statements 1)-3) for appropriate domains D_ε we use the following

Remark 24. Let f_ε be arbitrary (not necessarily nondegenerate) continuous deformation of a conformal map $f(t)=t+2\pi it^{k+1}(1+O(t))$, $\alpha_i(\varepsilon)$, $i=0,\dots,k$, be its continuous fixed point families (not necessarily distinct), $\alpha(\varepsilon)=\alpha_j(\varepsilon)$ be one of them. There exists a unique continuous in ε (including $\varepsilon=0$) polynomial vector field family

$$w_\varepsilon(t)=a(\varepsilon)\prod_{i=0}^{k}(t-\alpha_i(\varepsilon))$$

of degree $k+1$ with singularities at the fixed points of f_ε such that

$$f_\varepsilon\circ g_{w_\varepsilon}^{-1}(t)=t+O_1(t,\varepsilon), \tag{5}$$

$$O_1(t,\varepsilon)=O(w_\varepsilon(t)(t-\alpha(\varepsilon))),\ \ \frac{dO_1}{dt}(t,\varepsilon)=O(w_\varepsilon(t)(t-\alpha(\varepsilon)))+O(\frac{\partial(w_\varepsilon(t)(t-\alpha(\varepsilon)))}{\partial t}),$$

as $t,\varepsilon\to 0$. (Recall (Subsection 2.4) that for a vector field v by g_v^s we denote its time s flow map.) This follows from continuity of the family f_ε and is proved by straightforward calculation of the coefficient family $a(\varepsilon)$ we are looking for: $a(\varepsilon)=\prod_{s\neq j}(\alpha(\varepsilon)-\alpha_s(\varepsilon))^{-1}\ln(1+2\pi i\prod_{s\neq j}(\alpha(\varepsilon)-\alpha_s(\varepsilon)))$, in particular, $a(0)=2\pi i$. In the case, when the family f_ε is nondegenerate, the 1-jet of the perturbed field w_ε at its singular point $\alpha(\varepsilon)$ coincides with that of the canonic generator v_ε of the perturbed map f_ε.

The previous inequality 2) holds a priori in a neighborhood of $\alpha(\varepsilon)$ (that depends on ε) with $b=4\pi$ for small ε (the last statement of the previous Remark). To show

that it holds in a large domain, we construct a uniformly bounded family G_ε of positive continuous functions in appropriate f_ε^{-1}- invariant domains $D_\varepsilon \ni \alpha(\varepsilon)$, $G_\varepsilon(\alpha(\varepsilon)) = 2$, such that the tangent disc field $T_G = (|\dot{t}| < G_\varepsilon(t)|w_\varepsilon(t)|)$ is f_ε-invariant in D_ε. Then the inclusion $v_\varepsilon \in T_G$, which is valid in a neighborhood of $\alpha(\varepsilon)$ by construction and the last statement of the previous Remark, will be also valid in the orbits of f_ε in D_ε that go from $\alpha(\varepsilon)$ (f_ε- invariance of v_ε and T_G). Therefore, this inclusion will hold in the whole domain D_ε: the f_ε^{-1}- orbit of each point in D_ε converges to $\alpha(\varepsilon)$, by f_ε^{-1}- invariance of D_ε and lemma 2.2 in [10] (page 74). This together with the uniform boundedness of the functions G_ε will prove the inequality 2) for appropriate b independent on ε.

For the construction of the domains D_ε and the functions G_ε we consider the vector fields w_ε and a continuous in ε (including $\varepsilon = 0$) family $w_{0,\varepsilon} = e^{i\theta(\varepsilon)}w_\varepsilon$, $\theta(\varepsilon) \in \mathbb{R}$, of their constant multiples with the same moduli such that the perturbed fields $w_{0,\varepsilon}$ (correspondent to nonzero parameter values) have purely imaginary multipliers at their singularities $\alpha(\varepsilon)$ (so, the latters are centers). The possibility of choice of such a family continuous at $\varepsilon = 0$ follows from the convergence of the arguments of the multipliers at $\alpha(\varepsilon)$ of the fields w_ε. The latter follows from the convergence assumption for $\arg \alpha(\varepsilon)$ in the case, when $k = 1$, and the definition of nondegeneracy in the case, when $k \geq 2$. We prove the statements from the item preceding Remark 24 for the following domains D_ε.

Definition of the domains D_ε. Let $w_{0,\varepsilon}$ be a degree $k+1$ polynomial vector field family in $\mathbb{C}$ dependent continuously on the parameter ε (including $\varepsilon = 0$) with singularity families converging to 0, as $\varepsilon \to 0$. Let $\alpha(\varepsilon)$ be its continuous singularity family with purely imaginary (nonzero) multiplier for all $\varepsilon \neq 0$. Let U be a disc centered at zero in the phase plane. For any $\varepsilon \neq 0$ define R_ε to be the rotation basin of the singular point $\alpha(\varepsilon)$, which is the domain saturated by the closed trajectories of the field $w_{0,\varepsilon}$ surrounding $\alpha(\varepsilon)$. Define D_ε to be the maximal subdomain in $R_\varepsilon \cap U$ bounded by closed trajectory.

Example 4. In the conditions of the previous Definition let $k = 1$ (i.e., the vector field family $w_{0,\varepsilon}$ be quadratic). Then all the closed trajectories of the perturbed field $w_{0,\varepsilon}$ (and in particular the boundary of D_ε) are circles symmetric with respect to the line passing through its singularities. The rotation basin R_ε of the singular point $\alpha(\varepsilon)$ is the half-plane V_ε containing $\alpha(\varepsilon)$ and bounded by the line orthogonal and bisecting the segment joining the singularities. Thus, $D_\varepsilon \subset V_\varepsilon$ for any $\varepsilon \neq 0$. The line bounding V_ε has limit, as $\varepsilon \to 0$, which follows from the continuity of the family $w_{0,\varepsilon}$ at $\varepsilon = 0$ and the center assumption on $\alpha(\varepsilon)$. The disc D_ε tends to the disc D_0 contained in the limit half-plane $V = \mathrm{Lim}_{\varepsilon\to 0} V_\varepsilon$. Its boundary ∂D_0 is tangent to the boundary of V at 0 and to that of U (see Fig.5a,b). In particular, D_ε satisfy the statement 3) following the Definition of the domains Ω_ε from the beginning of the Subsection.

Fig.5a,b

Let us construct the function G_ε. Let R_ε be the rotation basin from the previous Definition. For $t \in R_\varepsilon$ define $L(t)$ to be the (closed) trajectory of the field $w_{0,\varepsilon}$ passing through t. Define the function $G'_\varepsilon(t)$ in $t \in R_\varepsilon$ to be equal to the maximal distance between $\alpha(\varepsilon)$ and a point of $L(t)$. Let $N > 0$. Put $G_\varepsilon = 2 + NG'_\varepsilon$. We show that for N large enough the correspondent function family G_ε is a one we are looking for (i.e., the correspondent tangent disc field $T_G: \ (|\dot{t}| < G_\varepsilon|w_\varepsilon|(t))$

is f_ε- invariant in D_ε, whenever U and ε are small enough). In the proof of this statement we use the following properties of the fields w_ε and $w_{0,\varepsilon}$.

Remark 25. Let f_ε be a nondegenerate family, $\alpha(\varepsilon)$ be its repelling fixed point family, w_ε, $w_{0,\varepsilon}$ be the correspondent vector field families from the item preceding the previous Definition. The fields w_ε and $w_{0,\varepsilon}$ commute. The angle between them is constant in t and bounded away from $\pi\mathbb{Z}$ uniformly in ε. The field w_ε has repelling singular point at $\alpha(\varepsilon)$. It is directed outside the domains bounded by closed trajectories of the field $w_{0,\varepsilon}$ surrounding $\alpha(\varepsilon)$, so these domains (and in particular, D_ε) are $-w_\varepsilon$- invariant. These statements follow from nondegeneracy of f_ε and the definitions of the families w_ε and $w_{0,\varepsilon}$. The correspondent function G'_ε (and hence, G_ε also) from the last item increases along the trajectories of the perturbed field w_ε. In particular, the correspondent tangent disc field T_G from the same item is w_ε- invariant (the last statement and w_ε- invariance of the field w_ε).

Firstly we prove Theorem 7 in the case, when $k = 1$. For larger k its proof is analogous and will be discussed at the end of the Subsection.

Let us prove that the domains D_ε are f_ε^{-1}- invariant, and moreover, so is any subdomain in D_ε bounded by closed trajectory of the field $w_{0,\varepsilon}$ (or equivalently, the function G'_ε increases along the orbits of f_ε in D_ε) whenever U and ε are small enough. Moreover, we prove the following estimate of growth of the function G'_ε along the f_ε- orbits: there exists a $b > 0$ such that the inequality

$$G'_\varepsilon(t) - G'_\varepsilon(f_\varepsilon^{-1}(t)) \geq b|w_\varepsilon(t)| \tag{6}$$

holds in D_ε whenever U and ε small enough. Further we use (6) in the proof of existence of invariant tangent disc field.

Firstly let us prove (6) for the maps $g_{w_\varepsilon}^{-\frac{1}{2}}$ instead of f_ε^{-1}. The function G'_ε decreases along the orbits of $g_{w_\varepsilon}^{-\frac{1}{2}}$ (the previous Remark). We use the following estimate of the distance of a point t to the $g_{w_\varepsilon}^{-\frac{1}{2}}$- image of the correspondent curve $L(t)$: for any bounded neighborhood U of zero in the t- line there exists a $b > 0$ such that for any ε small enough and $t \in R_\varepsilon \cap U$

$$\mathrm{dist}(t, g_{w_\varepsilon}^{-\frac{1}{2}}(L(t))) \geq b|w_\varepsilon(t)|. \tag{6'}$$

Let us prove (6'). By $\beta(\varepsilon)$ denote the angle between w_ε and $w_{0,\varepsilon}$ (which is bounded away from $\pi\mathbb{Z}$ by the previous Remark). We use the fact that the domain $M(t,\varepsilon) = \{g_{w_\varepsilon}^s(t), \ \ |s| < \frac{1}{2}\sin\beta(\varepsilon)\}$ is disjoint from the curve $g_{w_\varepsilon}^{-\frac{1}{2}}(L(t))$. This follows from the definition of $\beta(\varepsilon)$ and the fact that the fields w_ε and $w_{0,\varepsilon}$ commute. (In fact, the boundary of the domain $M(t,\varepsilon)$ touches this curve.) Let $0 < b < \frac{1}{2}\inf\sin\beta(\varepsilon)$. The domain $M(t,\varepsilon)$ contains $b|w_\varepsilon(t)|$- neighborhood of t, whenever ε and t are small enough, since $g_{w_\varepsilon}^s(t) - t = sw_\varepsilon(t)(1 + o(1))$, as $|s| \leq 1$ and $t, \varepsilon \to 0$. This together with the previous statement proves (6').

Now let us prove inequality (6) for the maps $g_{w_\varepsilon}^{-\frac{1}{2}}$. Recall that the restriction of the function G'_ε to the trajectories of the field $w_{0,\varepsilon}$ is constant. Let $t \in R_\varepsilon$. The curves $L(t)$ and $g_{w_\varepsilon}^{-\frac{1}{2}}(L(t))$ are circles surrounding $\alpha(\varepsilon)$ symmetric with respect to the line passing through the singularities of the field $w_{0,\varepsilon}$ (Example 4). Let τ be the point of the second curve having maximal distance to $\alpha(\varepsilon)$. Then τ

lies in the same line and is separated from 0 by $\alpha(\varepsilon)$ in the line. By definition, $G'_\varepsilon|_{g_{w_\varepsilon}^{-\frac{1}{2}}(L(t))} \equiv |\tau - \alpha(\varepsilon)|$. Let τ' be the point of the curve $L(t)$ that lies in this line on the same side from $\alpha(\varepsilon)$, as τ. Then τ' is the point of the curve $L(t)$ of maximal distance to $\alpha(\varepsilon)$ (so, $G'_\varepsilon(t) = |\tau' - \alpha(\varepsilon)|$). Therefore,

$$G'_\varepsilon(t) - G'_\varepsilon(g_{w_\varepsilon}^{-\frac{1}{2}}(t)) = |\tau' - \alpha(\varepsilon)| - |\tau - \alpha(\varepsilon)| = |\tau' - \tau| \geq b|w_\varepsilon(\tau')|.$$

The last inequality holds whenever the curve $L(t)$ is close enough to 0 and ε is small enough, i.e., the initial point t lies in the domain D_ε correspondent to U and ε small enough (estimate (6')). Now estimate (6) for the maps $g_{w_\varepsilon}^{-\frac{1}{2}}$ follows from the last inequality and the fact that the restriction $|w_\varepsilon||_{L(t)}$ takes its maximal value at τ' (since so does $|p(t,\varepsilon)||_{L(t)}$ by definition).

Now let us prove (6) for the maps f_ε^{-1}. To do this, we use the estimate (6) for the maps $g_{w_\varepsilon}^{-\frac{1}{2}}$ proved before, estimate (6') for the maps $g_{w_\varepsilon}^{\frac{1}{2}}$ (which is valid, in fact, for any fixed real time w_ε- flow map and is proved in the same way, as for the time value $-\frac{1}{2}$ above) and the asymptotic formula $f_\varepsilon^{-1}(t) - g_{w_\varepsilon}^{-1}(t) = o(w_\varepsilon(t)) = o(w_\varepsilon(g_{w_\varepsilon}^{-1}(t)))$, as $t, \varepsilon \to 0$ (formula (5)). Then for any U and ε small enough for any $t \in D_\varepsilon$ the image $f_\varepsilon^{-1}(L(t))$ is contained in the domain bounded by the curve $g_{w_\varepsilon}^{-\frac{1}{2}}(L(t))$. This follows from estimate (6') applied to the map $g_{w_\varepsilon}^{\frac{1}{2}}$ at the point $g_{w_\varepsilon}^{-1}(t)$ and the last asymptotic formula. Then

$$G'_\varepsilon(t) - G'_\varepsilon(f_\varepsilon^{-1}(t)) \geq G'_\varepsilon(t) - G'_\varepsilon(g_{w_\varepsilon}^{-\frac{1}{2}}(t)).$$

This together with the inequality (6) for the maps $g_{w_\varepsilon}^{-\frac{1}{2}}$ proves the same inequality for the maps f_ε. Estimate (6) is proved.

Now let us show that for any U, ε small enough and N large enough the tangent disc field T_G is f_ε- invariant in D_ε. To do this, we use (6) and the following estimate of the difference between the field w_ε and its f_ε- image: there exists a $d > 0$ such that for any U, ε small enough the inequality

$$|(f_\varepsilon)_* w_\varepsilon - w_\varepsilon| \leq d|w_\varepsilon|^2 \tag{7}$$

holds in D_ε. In the proof of this estimate we use the following statement:

$(*)$ for any ε and $t \in R_\varepsilon$ $\alpha(\varepsilon)$ is the closest to t of the singularities of the field w_ε.

By (5) and $g_{w_\varepsilon}^{-1}$- invariance of w_ε,

$$\begin{aligned}((f_\varepsilon)_* w_\varepsilon - w_\varepsilon)(t) &= ((f_\varepsilon \circ g_{w_\varepsilon}^{-1})_* w_\varepsilon - w_\varepsilon)(t) \\ &= O((w_\varepsilon(t))^2) + O(w_\varepsilon(t)\frac{\partial((t-\alpha(\varepsilon))w_\varepsilon(t))}{\partial t}).\end{aligned}$$

The last term in the right-hand side of this formula is $O((w_\varepsilon(t))^2) + O((t-\alpha(\varepsilon))^2 w_\varepsilon(t))$, as $t, \varepsilon \to 0$. The second term in the last expression is $O((w_\varepsilon(t))^2)$, as $t \in R_\varepsilon$, $\varepsilon \to 0$ (statement $(*)$). This proves (7).

For the proof of invariance of the disc field T_G for large N let us calculate and estimate in D_ε the difference of the radius of disc of the field T_G and that of disc of its image $(f_\varepsilon)_* T_G$. By definition, this difference is equal to

$$\begin{aligned}&(2 + NG'_\varepsilon)|w_\varepsilon| - (2 + NG'_\varepsilon \circ f_\varepsilon^{-1})|(f_\varepsilon)_* w_\varepsilon| \\ &= N(G'_\varepsilon - G'_\varepsilon \circ f_\varepsilon^{-1})|w_\varepsilon| - (2 + NG'_\varepsilon \circ f_\varepsilon^{-1})(|(f_\varepsilon)_* w_\varepsilon| - |w_\varepsilon|).\end{aligned} \tag{8}$$

It suffices to show that this difference is positive in D_ε, whenever N is large enough and U, ε are small enough. The first term of the difference in the right-hand side of (8) is greater than $Nb|w_\varepsilon|^2$ by (6). The module of the second term is not greater than $d|w_\varepsilon|^2(2+NG'_\varepsilon\circ f_\varepsilon^{-1})$ by (7). The latter can be made arbitrarily smaller with respect to the former ($Nb|w_\varepsilon|^2$) (and hence, to the first term in the right-hand side of (8)) in D_ε, if N be chosen large enough and U, ε small enough. Indeed the first term in its brackets is a constant independent on N and ε; the second term is less than N times the radius of the disc U (which is greater than $G'_\varepsilon|_{D_\varepsilon}$ by definition). Thus, the second term in (8) can be made less than $Nb|w_\varepsilon|^2$, then the difference (8) will be positive. The invariance of the tangent disc field is proved.

Now for the proof of Theorem 7 in the case, when $k=1$, it suffices to prove the statement 1) following the Definition of the domains Ω_ε from the beginning of the Subsection. Let V be the limit half-plane from Example 4. The statement 1) holds for any sector C whose closure punctured at 0 is contained in V. The sector V is good: it contains the unique imaginary dividing ray, the latter lies in S by definition. Therefore, the sector C as above may be chosen to contain this ray as well and to be contained in S (then it will be good). Theorem 7 is proved in the case, when $k=1$.

Now let us prove Theorem 7 in the case, when $k\geq 2$. Its proof in the previous case remains valid for larger k modulo the statements 1) and 3) following the Definition of the domains Ω_ε from the beginning of the Subsection, estimate (6) and statement $(*)$ from the item following (7). In the proof of these statements we use the following

Proposition 3. *Let $k\geq 2$, $w_{0,\varepsilon}$ be a family of degree $k+1$ polynomial vector fields in complex line with the coordinate t depending continuously on the parameter ε such that the (nonperturbed) field $w_{0,0}$ is a constant multiple of the monomial t^{k+1} and the perturbed field (correspondent to nonzero parameter value) has $k+1$ singularities that form continuous families $\alpha_i(\varepsilon)$, $i=0,\dots,k$, satisfying the asymptotic polygon regularity statement from Remark 10. Let Δ be the correspondent asymptotic regular polygon. Let $\alpha(\varepsilon)$ be a center singularity family (i.e., the correspondent multipliers be purely imaginary whenever $\varepsilon\neq 0$), A be the correspondent vertex of Δ. Let V (V_ε) be the radial sector with the angle $\frac{\pi}{k}$ containing A and symmetric with respect to the radial line of A (respectively, $\alpha(\varepsilon)$). Let W_ε be the sector containing $\alpha(\varepsilon)$ bounded by the radial rays orthogonal to the singularity polygon sides neighboring at $\alpha(\varepsilon)$. For any ε small enough the rotation basin R_ε of the center singularity $\alpha(\varepsilon)$ is contained in W_ε and bounded by the trajectory of the field $w_{0,\varepsilon}$ that goes from infinity and returns to it in the asymptotic directions of the boundary rays of V_ε (not necessarily approaching the latters). The rotation basin family R_ε converges to V, as $\varepsilon\to 0$ (see Fig.6 a,b in the case, when $k=2$). For any neighborhood U of zero in the phase line there exists an $r'>0$ such that the correspondent domains D_ε from the previous Definition satisfy the statement 3) following the Definition of the domains Ω_ε at the beginning of the Subsection.*

Fig. 6a,b

Corollary 5. *In the conditions of the previous Proposition let U be a neighborhood of zero in the t- line, D_ε be the correspondent family of domains from the previous Definition. The family D_ε converges to a subdomain $D_0\subset U$ bounded by the*

trajectory of the field $w_{0,0}$ that goes from zero and returns to it with asymptotic tangency to the boundary rays of V and touches the boundary of U (see Fig.6b in the case, when $k = 2$).

Proof of Proposition 3. The proofs of the statements of Proposition 3 on the boundary of R_ε and the convergence of the latter are implicitly contained in subsection 6 of [6]. Let us show that for any ε small enough $R_\varepsilon \subset W_\varepsilon$. To do this, we show that for any ε small enough the field $w_{0,\varepsilon}$ is transversal to the boundary rays of W_ε. Indeed in the case, when the singularity polygon is regular and centered at 0, the field $w_{0,\varepsilon}$ has constant nonzero angles with the boundary rays of W_ε: the correspondent real tangent line field is constant in the union of the symmetry lines of the polygon (subsection 6 in [6]); therefore, it is orthogonal to the radial ray of the singular point $\alpha(\varepsilon)$ (the latter is center); hence, its angles with the boundary rays of the sector W_ε are constant and equal to $\frac{\pi}{2} - \frac{\pi}{k+1} > 0$. Now for the proof of transversality in the general case we consider the new vector field family $\tilde{w}_\varepsilon$ obtained from w_ε by the radial homothethy family from Remark 10, which transforms the singularity polygon of the latter to a polygon with unit diameter converging to the correspondent regular polygon Δ. The transversality statement from the beginning of the item is equivalent to that for the new family $\tilde{w}_\varepsilon$. This statement for $\tilde{w}_\varepsilon$ follows from continuity in ε of the correspondent real tangent line field and the same statement for the limit field $\tilde{w}_0$, which has a regular singularity polygon.

We prove the inclusion $R_\varepsilon \subset W_\varepsilon$ by contradiction, using only the previous transversality statement. Suppose $R_\varepsilon \not\subset W_\varepsilon$. Then there exists a closed trajectory of the field $w_{0,\varepsilon}$ in R_ε tangent to a boundary ray of W_ε, since, by assumption, there exists at least one closed trajectory in W_ε ($\alpha(\varepsilon) \in W_\varepsilon$). This contradicts the transversality statement. The inclusion statement of Proposition 3 is proved.

Now let us prove the last statement of Proposition 3. It is reduced to its particular case, when the singularity polygons are regular, analogously to the discussion from the item following Proposition 3. In this case for all ε the domain D_ε is symmetric with respect to the radial line of $\alpha(\varepsilon)$. Its intersection with this line is connected: otherwise D_ε, which is simply connected by definition, would not be so by symmetry. The last statement of Proposition 3 follows from the convergence of this intersection to that of the limit line with the limit domain D_0 from Corollary 5, which is a nonempty radial interval of the limit of the radial ray of $\alpha(\varepsilon)$. Proposition 3 is proved.

Now let us prove the four statements from the item preceding Proposition 3. The statement 3) follows from the last statement of Proposition 3. Let us prove the statement 1). The inclusion $D_0 \supset C^r$ holds for any sector C whose closure with zero deleted is contained in the sector V from Proposition 3 and appropriate $r > 0$ depending on C (Corollary 5). By definition (Remark 11), the sector V is contained in S and is good (contains the same imaginary dividing ray, as S). Therefore, the sector C as above may be chosen to be good as well.

Let us prove statement ($*$). Let us consider that $\alpha(\varepsilon)$ is the singularity closest to 0 for all ε: one can achieve this by applying appropriate continuous translation family in the t- line so that the asymptotic regularity statement from Remark 10 will remain valid. Then ($*$) follows from Proposition 3 (the inclusion $R_\varepsilon \subset W_\varepsilon$ and the definition of the sector W_ε).

Let us prove estimate (6). Its prove is done in the same way, as in the previous case modulo the same inequality for the maps $g_{w_\varepsilon}^{-\frac{1}{2}}$ instead of f_ε^{-1}. Let us prove

this inequality. To do this, we use estimate (6'), which is valid for any fixed real time flow map of the field w_ε and is proof repeats that in the previous case. For $t \in R_\varepsilon$ by $\tau(t)$ ($\tau'(t)$) denote the point of the curve $g_{w_\varepsilon}^{-\frac{1}{2}}(L(t))$ (respectively, $L(t)$) that has the maximal distance to $\alpha(\varepsilon)$. By definition, the functions $\tau'(t)$ and $\tau(t)$ are constant in the trajectories of the field $w_{0,\varepsilon}$. Then

$$G'_\varepsilon|_{L(t)} - G'_\varepsilon|_{g_{w_\varepsilon}^{-\frac{1}{2}}(L(t))} \geq \mathrm{dist}(\tau(t),\ L(t)) \geq b|w_\varepsilon(\tau(t))|$$

(the first inequality follows from the definition of the functions G'_ε, τ and τ'; the last inequality follows from the estimate (6') applied to the map $g_{w_\varepsilon}^{\frac{1}{2}}$ at the point $\tau(t)$). Now for the proof of (6) for the maps $g_{w_\varepsilon}^{-\frac{1}{2}}$ it suffices to show that there exists a $c > 0$ such that for any U, ε small enough and $t \in D_\varepsilon$

$$|w_\varepsilon(t)| \leq c|w_\varepsilon(\tau(t))|. \tag{9}$$

In the proof of the last inequality we use statement $(*)$ proved before and the fact that for any U, ε small enough and $t \in D_\varepsilon$

$$|t - \alpha(\varepsilon)| \leq 2|\tau(t) - \alpha(\varepsilon)| = 2G'_\varepsilon|_{g_{w_\varepsilon}^{-\frac{1}{2}}(L(t))}. \tag{10}$$

The latter follows from the inequality $|t - \alpha(\varepsilon)| \leq G'_\varepsilon|_{L(t)}$ and the asymptotic formula $G'_\varepsilon|_{L(t)} = G'_\varepsilon|_{g_{w_\varepsilon}^{-\frac{1}{2}}(L(t))}(1 + o(1))$, as $\varepsilon, L(t) \to 0$ (the asymptotic formula $t' = g_{w_\varepsilon}^{-\frac{1}{2}}(t')(1 + o(1))$, as $\varepsilon, t' \to 0$).

For the proof of (9) it suffices to show that for any other singularity family $\alpha_i \neq \alpha$ $|t - \alpha_i(\varepsilon)| \leq 4|\tau(t) - \alpha_i(\varepsilon)|$, whenever U, ε are small enough and $t \in D_\varepsilon$. This is implied by the following inequalities:

$$|t - \alpha_i| \leq |t - \alpha| + |\alpha - \alpha_i|;\ \ |t - \alpha| \leq 2|\tau(t) - \alpha| < 2|\tau(t) - \alpha_i|;$$

$$|\alpha - \alpha_i| \leq |\tau(t) - \alpha| + |\tau(t) - \alpha_i| < 2|\tau(t) - \alpha_i|.$$

The inequalities in the first line follow from the triangle inequality, (10) and statement (*) respectively. Those in the second line follow from the triangle inequality and (*) respectively. Estimate (9) (and hence, (6) also) is proved. The proof of Theorem 7 is completed.

3.5. Proof of the two last statements of Proposition 2.

In the conditions of Proposition 2 consider the new family $w_{0,\varepsilon} = e^{i\theta(\varepsilon)}w(\varepsilon)$, $\theta(\varepsilon) \in \mathbb{R}$, of multiples of the fields $w(\varepsilon)$ having purely imaginary multipliers at $\alpha(\varepsilon)$. Without loss of generality we consider that the argument of the singularity $\alpha(\varepsilon)$ has limit, as $\varepsilon \to 0$. In the case, when $k \geq 2$, this condition holds by definition. In the case, when $k = 2$, one can achieve this by taking appropriate subsequence of arbitrary convergent sequence $\varepsilon_n \to 0$ instead of all the parameter space. Then the family $w_{0,\varepsilon}$ can be chosen to be continuous at $\varepsilon = 0$. Let D_ε be the family of domains from the previous Definition correspondent to the neighborhood U_t. Then $D_\varepsilon \subset \Omega_\varepsilon$ by definition. This together with the last statement of Proposition 3 proves the last statement of Proposition 2. Let $k \geq 2$. Let us prove that the connected component of the intersection $\Omega_\varepsilon \cap \Omega$ containing $\alpha(\varepsilon)$ tends to Ω, as

$\varepsilon \to 0$. This will prove Proposition 2. To do this, it suffices to show that just some subdomain of the intersection converges to Ω: the last statement of Proposition 2 will imply that this holds for the intersection component containing $\alpha(\varepsilon)$, as in Subsection 3.2. Let D_0 be the limit domain from the previous Corollary. Then $D_0 \subset \Omega$. Let C be a good sector, $r > 0$, such that $C^r \subset D_0$. We consider that C contains the repelling ray correspondent to the limit vector field $w(0)$ (one can achieve this by choosing C large enough). Then a connected component of the intersection $D_\varepsilon \cap C^r$ (and hence, $\Omega_\varepsilon \cap C^r$) converges to C^r. Consider the family of domains saturated by the trajectories of the fields $w(\varepsilon)$ in Ω that start in $C^r \cap \Omega_\varepsilon$. This family converges to Ω by definition. The proof of the two last statements of Proposition 2 is completed.

3.6. Another proof of Theorem 3(4) that uses Theorem 7.

Let us sketch another proof of Theorems 3 and 4. Let $D = \{|t| < \delta, z = \delta\}$ be a disc transversal to the phase curves of the vector fields from the family $(1)_\varepsilon$. Consider the monodromy map family f_ε of the fields $(1)_\varepsilon$ in D correspondent to going around the singular point 0 in the separatrix $t = 0$ of the nonperturbed field. The family f_ε is nondegenerate in the sense of the Definition from Subsection 2.4.C. (In fact, the nondegeneracy of the monodromy family is equivalent to that of the family $(1)_\varepsilon$ under considerations.)

As it is shown below, Theorems 3 and 4 are implied by Theorem 7 and the two following Remarks.

Remark 26. Let (1) be a vector field as in Section 1 holomorphic in a neighborhood $U = \{|z| < 2\delta\} \times \{|t| < \delta\}$ of zero. Let $D = \{z = \delta\} \cap U$ be a disc transversal to (1) in the line $z = \delta$ equipped with the coordinate t. Then the monodromy map of (1) in D correspondent to the counterclockwise going around zero in the separatrix $t = 0$ is holomorphic in a neighborhood of the point $t = 0$ in D and satisfies the conditions of Theorem 5. Let S be a good sector in the t- axis, v be the correspondent sectorial canonic generator of the monodromy map, τ be its complex time. The correspondent sectorial canonic first integral of (1) restricted to D is a constant multiple of the function $e^{2\pi i\tau}$. This follows from the same statement for the formal normal form $(1)_n$ and Theorem 1.

Remark 27. Let

$$\begin{cases} \dot{z} = \nu z \\ \dot{t} = \mu t \end{cases}$$

be a linear vector field with $\frac{\mu}{\nu} \notin \mathbb{R}$, $|\frac{\mu}{\nu}| < \frac{1}{2}$. Let $D = \{|t| < \delta, \ z = \delta\}$ be a disc transversal to the field equipped with the coordinate t. Let M be the monodromy map of the field in D correspondent to going around zero in the separatrix $t = 0$. The multiplier of M at its fixed point $t = 0$ is equal to $\lambda = e^{2\pi i\frac{\mu}{\nu}}$. The canonic first integral of the vector field and the canonic generator of M at the fixed point are well-defined ($|\lambda| \neq 1$). Let τ be the (multivalued) complex time in D correspondent to the generator. The restriction of the canonic integral to D (which is a multivalued function branched at $t = 0$) is a constant multiple of the function $e^{2\pi i\tau}$. This statement remains valid for a nonlinear vector field having a singular point with multipliers as at the beginning of the item. This follows from Theorem 2.

Now let us sketch the proof of the convergence statement of Theorem 3(4). The restrictions of appropriately normalized canonic first integrals of $(1)_\varepsilon$ to the transversal disc D converge to that of the correspondent sectorial canonic first

integral. This follows from the two previous Remarks and Theorem 7 applied to the monodromy family f_ε. Now for the proof of Theorem 3(4) it suffices to show that the canonic integrals of the perturbed fields extend analytically to domains satisfying statement 1) of Theorem 3 (along the phase curves, where they are constant), and this extension depends continuously on the parameter. To do this, one shows that for appropriately chosen neighborhoods U_z, U_t of zero in the coordinate lines and appropriate domains $\Omega_\varepsilon \subset U_t$ (e.g., the subdomains Ω_ε^2 from Subsection 3.3) each point of the domain $\widetilde{\Omega_\varepsilon} = U_z \times \Omega_\varepsilon$ can be connected in the latter to a point of D where the monodromy generator is well-defined by path lying in the phase curve of $(1)_\varepsilon$, and one can choose these paths to be dependent continuously on the parameter. The details of the proof are omitted to save the space.

3.7. Proof of Lemma 1.

Let

$$(2)_\varepsilon \qquad \begin{cases} \dot{z} = f(z,t,\varepsilon) \\ \dot{t} = g(z,t,\varepsilon) \end{cases}$$

be arbitrary continuous deformation of a vector field (1). This means that the functions f and g are continuous in ε and holomorphic in (z,t), $f(z,t,0) = z + O(|z|^2 + |t|^{k+1})$, $g(z,t,0) = t^{k+1}$. We show that $(2)_\varepsilon$ is orbitally analytically equivalent to a family $(1)_\varepsilon$, i.e., phase curves of the fields $(2)_\varepsilon$ can be transformed to those of $(1)_\varepsilon$ by continuous family of analytic changes of the variables (z,t) defined for all the parameter values small enough. This will prove Lemma 1.

Without loss of generality we assume that all the singular points of the vector fields $(2)_\varepsilon$ lie in the t- axis. One can achieve this by applying the family of coordinate changes of the variable z defined by the formula $\tilde{z} = f(z,t,\varepsilon)$. Then the family of vector fields under consideration is of the type

$$(3)_\varepsilon \qquad \begin{cases} \dot{z} = z(1+q(z,t,\varepsilon)) + g(t,\varepsilon)\prod_{i=0}^{k}(t-\alpha_i(\varepsilon)) \\ \dot{t} = \prod_{i=0}^{k}(t-\alpha_i(\varepsilon)) + zR(z,t,\varepsilon) \end{cases}$$

(up to multiplication by family of nonzero holomorphic functions), where the functions q, g, R are holomorphic in (z,t) and continuous in the parameter ε, as are the constants $\alpha_i(\varepsilon)$, $\alpha_i(0) = 0$, $q(0,0,0) = 0$, $R(z,t,0) \equiv 0$. The singular points of these vector fields are $(0, \alpha_i(\varepsilon))$. Let us show that there exists a family of changes of the variable t that transforms $(3)_\varepsilon$ to an analogous family with $R \equiv 0$. This will prove Lemma 1.

Let $\lambda_i(\varepsilon)$ be the family of eigenvalues of the linearization operator of $(3)_\varepsilon$ at the continuous singular point family $(0, \alpha_i(\varepsilon))$ such that the correspondent eigenline family approaches the line tangent to the z- axis, as $\varepsilon \to 0$ (so, $\lambda_i(0) = 1$). Let $\mu_i(\varepsilon)$ be the other eigenvalue family. Then $\mu_i(0) = 0$. For any ε small enough there exists a unique separatrix $L_i(\varepsilon)$ passing through the singular point $(0, \alpha_i(\varepsilon))$ and tangent to the eigenline of the linearization operator at $(0, \alpha_i(\varepsilon))$ correspondent to the eigenvalue $\lambda_i(\varepsilon)$. In general, for any planar vector field in a neighborhood of its singular point with multipliers λ and μ, $|\lambda| > |\mu|$, there exists a unique regular holomorphic curve tangent to the field and passing through the singularity with tangency to the λ- eigenline of the linearization operator. In the case, when $\frac{\mu}{\lambda} \notin \mathbb{R}_+$, this follows from either Theorem 2 (when this ratio is not real), or the analytic version of Hadamard - Perron theorem [12] (when it is either real negative, or zero). In the case, when this ratio is positive, the existence and uniqueness of

separatrix statement is reduced to the previous one by applying blowing-up. The last statement from the last item is equivalent to the existence of a t- coordinate change family that rectifies all these separatrices simultaneously. To prove this its reformulation, we use the convergence of the separatrices of the perturbed field to that of the nonperturbed field. This is implied by the following

Proposition 4. *Let $(3)_\varepsilon$ be as at the beginning of the Subsection and in the item following its formula, $L_i(\varepsilon)$ be a separatrix family from the last item. There exist a neighborhood $U = U_z \times U_t$ of zero in the phase space such that for any ε small enough the local separatrices $L_i(\varepsilon) \cap U$ are graphs $t = F_{i,\varepsilon}(z)$ of functions $F_{i,\varepsilon}(z)$ holomorphic in U_z and depending continuously on the parameter (including $\varepsilon = 0$, where $F_{i,0} \equiv 0$) such that $|(F_{i,\varepsilon})'_z| \leq 1$, $|F_{i,\varepsilon}(z)| \leq |z|$.*

Proof. Let $(0, \alpha_i(\varepsilon))$ be a continuous family of singularities of $(3)_\varepsilon$. Let K and $\widetilde{K}$ be respectively the tangent cone field and the cone from the proof of Lemma 4 (Subsection 3.2). The inequalities of Proposition 4 are equivalent to the statement that the tangent lines to $L_i(\varepsilon)$ lie outside K and $L_i(\varepsilon)$ lies outside $\widetilde{K}$. These inclusions hold a priori in a neighborhood of $(0, \alpha_i(\varepsilon))$ (that depends on ε). As in the proof of Lemma 2, to show that they hold in a large domain, we use the fact that there exists a neighborhood U of zero where the field of the complements to the cones K (and hence, the complement to $\widetilde{K}$) are invariant with respect to the field $(3)_\varepsilon$ for all ε small enough (proposition 2 in [6]). Let us consider that $U = U_z \times U_t$, where $U_z = \{|z| < \delta\}$, $U_t = \{|t| < 2\delta\}$. Let $L'_i(\varepsilon) \subset L_i(\varepsilon) \cap U$ be the subdomain saturated by the trajectories of the field $(3)_\varepsilon$ in $L_i(\varepsilon) \cap U$ that go from $(0, \alpha_i(\varepsilon))$. Then $TL'_i \cap K = \emptyset$, $L'_i \cap \widetilde{K} = \emptyset$, by invariance of the cone K complement field. Now for the proof of Proposition 4 (modulo continuity of the dependence on the parameter) it suffices to show that $L'_i(\varepsilon)$ is the graph $t = F_{i,\varepsilon}(z)$ of a holomorphic function in U_z (then $L'_i(\varepsilon) = L_i(\varepsilon) \cap U$). This is equivalent to the statement that $L'_i(\varepsilon)$ is 1-to-1 projected onto the disc U_z in the z- line. This is implied by the following properties of this projection: 1) it does not have critical points (the tangent lines to $L'_i(\varepsilon)$ are not parallel to the t- line, since they lie outside K); 2) it is a map "onto" U_z. The last statement follows from the fact that the trajectories that form $L'_i(\varepsilon)$ meet the boundary of U at points whose projection images in the z- line lie in the boundary of the disc U_z. This follows from the inclusion $L'_i(\varepsilon) \subset U \setminus \widetilde{K}$ and the definition of U (in the same way, as in the proof of lemma 1 from [6]). The inequalities of Proposition 4 are proved. Let us prove the continuity of the family of the correspondent functions $F_{i,\varepsilon}$ in ε, e.g., at $\varepsilon = 0$ (for $\varepsilon \neq 0$ the proof is analogous). The family $F_{i,\varepsilon}$ converges to 0 uniformly in compact subsets of U_z, as $\varepsilon \to 0$. This follows from its normality (inequalities of Proposition 4) and the uniqueness of the separatrix of the nonperturbed field transversal to the t- axis, as in the proof of Lemma 2 in Subsection 3.2. Proposition 4 is proved.

By Proposition 4, each family $L_i(\varepsilon)$ is continuous in the parameter, and hence, can be rectified itself (i.e., transformed to a family of subdomains of complex lines parallel to the z- axis) by a continuous family of changes of the variable t. Now we show that these families can be rectified simultaneously by t- variable changes depending continuously on the parameter. This will prove Lemma 1.

Without loss of generality we consider that the singular point family $(0, \alpha_0(\varepsilon))$ is identically zero, and the correspondent separatrices $L_0(\varepsilon)$ already lie in the z- axis. This means that the family $(3)_\varepsilon$ under considerations is of the type

$$
(4)_\varepsilon \qquad \begin{cases} \dot z = z(1+q(z,t,\varepsilon)) + tg(t,\varepsilon)\prod_{i=1}^k (t-\alpha_i(\varepsilon)) \\ \dot t = t(\prod_{i=1}^k (t-\alpha_i(\varepsilon)) + zR(z,t,\varepsilon)), \end{cases}
$$

Let us firstly show that there exists a continuous family of changes of the variable t that preserves the line $t=0$ (in particular, the separatrix $L_0(\varepsilon)$) and the singularity family $(0,\alpha_1(\varepsilon))$ and transforms the correspondent separatrix $L_1(\varepsilon)$ to a subdomain of the line $t=\alpha_1(\varepsilon)$. Let $F_{1,\varepsilon}$ be the correspondent function from Proposition 4. The family of changes we are looking for is $\tau = t\frac{\alpha_1(\varepsilon)}{F_{1,\varepsilon}(z)}$. By construction, it preserves $L_0(\varepsilon)$ and rectifies $L_1(\varepsilon)$. Let us prove that these changes form a continuous family. To do this, it suffices to show that they converge to the identity, as $\varepsilon\to 0$.

For the proof of the last statement we show that

$$
(\ln F_{1,\varepsilon})'_z = \frac{(F_{1,\varepsilon})'_z}{F_{1,\varepsilon}} \to 0, \text{ as } \varepsilon \to 0
$$

uniformly in compact sets. This together with the equality $F_{1,\varepsilon}(0)=\alpha_1(\varepsilon)$ will imply that $\frac{F_{1,\varepsilon}(z)}{\alpha_1(\varepsilon)} \to 1$, and hence, the convergence of the coordinate changes under considerations to the identity. The calculation of the derivative of the function $F_{1,\varepsilon}$ yields

$$
F_{1,\varepsilon}(z)'_z = \frac{dt}{dz}|_{L_1(\varepsilon)} = \frac{t((t-\alpha_1(\varepsilon))\prod_{i=2}^k(t-\alpha_i(\varepsilon)) + R(z,t,\varepsilon)z)}{z(1+q(z,t,\varepsilon)) + g(t,\varepsilon)t(t-\alpha_1(\varepsilon))\prod_{i=2}^k(t-\alpha_i(\varepsilon))}|_{t=F_{1,\varepsilon}(z)}.
$$

Therefore,

$$
(\ln F_{1,\varepsilon}(z))'_z = \frac{\frac{(t-\alpha_1(\varepsilon))}{z}\prod_{i=2}^k(t-\alpha_i(\varepsilon)) + R(z,t,\varepsilon)}{1+q(z,t,\varepsilon)+g(t,\varepsilon)\frac{t-\alpha_1(\varepsilon)}{z}t\prod_{i=2}^k(t-\alpha_i(\varepsilon))}|_{t=F_{1,\varepsilon}(z)}.
$$

The ratio $\frac{t-\alpha_1(\varepsilon)}{z}|_{t=F_{1,\varepsilon}(z)}$ in the right-hand side of the last formula has module at most 1 (the last inequality of Proposition 4). Therefore, there exists a neighborhood U_z of zero in the z -axis (independent on ε) where the right-hand side has module less or equal to $2(\prod_{i=2}^k(|F_{1,\varepsilon}(z)|+|\alpha_i(\varepsilon)|)+|R(z,F_{1,\varepsilon}(z),\varepsilon)|)$ for all ε small enough. The last expression converges to 0, as $\varepsilon\to 0$, since so do $F_{1,\varepsilon}(z)$, $\alpha_i(\varepsilon)$ and $R(z,t,\varepsilon)$. The statements on the convergence of the coordinate changes to the identity is proved. (This proves the simultaneous rectification statements for the separatrices in the case, when $k=1$.)

We have already proved the possibility to rectify two separatrix families simultaneously by continuous family of changes of the t- coordinate. The possibility to rectify 3 or more separatrix families simultaneously is proved analogously by induction in their number. For example, let us show that if 2 separatrix families (say, $L_0(\varepsilon)$ and $L_1(\varepsilon)$) are already rectified (i.e., lie in lines $t=\alpha_0(\varepsilon)(=0)$, $t=\alpha_1(\varepsilon)$ respectively), then there exists a continuous family of changes of the t- coordinate that preserves them and rectifies another separatrix family (say, $L_2(\varepsilon)$).

The rectifying family we are looking for is the family of changes

$$
\tau = t - t(t-\alpha_1(\varepsilon))\frac{F_{2,\varepsilon}(z)-\alpha_2(\varepsilon)}{F_{2,\varepsilon}(z)(F_{2,\varepsilon}(z)-\alpha_1(\varepsilon))}.
$$

As before, it suffices to prove the convergence of this family to identity, as $\varepsilon \to 0$. To do this in its turn, it suffices to show that $\ln(F_{2,\varepsilon})'_z = o(\alpha_2(\varepsilon)-\alpha_1(\varepsilon))$, as $\varepsilon \to 0$. The proof of this statement repeats that of the analogous statement for $F_{1,\varepsilon}$ proved before with obvious changes. Lemma 1 is proved.

4. A higher-dimensional central manifold analogue of Theorems 3 and 4

4.1. Statement of results. Existence and uniqueness of sectorial central manifolds of general higher-dimensional saddle-node vector fields. Their expression as limits of separatrices of generic perturbation.

In this Subsection we consider saddle-node singularities of higher dimensional holomorphic vector fields, which are defined in exactly the same way, as in the two-dimensional case (at the beginning of the paper).

Remark 28. Any vector field with a saddle-node singularity is locally orbitally analytically equivalent to a field of the type

$$\begin{cases} \dot{y} = By + O(|y|^2 + |t|^{k+1}) \\ \dot{t} = t^{k+1} + O(|y|^2) \end{cases} \tag{11}$$

in a neighborhood of zero in $\mathbb{C}^n$ with the coordinates $(y = (y_1, \dots, y_{n-1}),\ t)$, where B is a nondegenerate upper-triangular block-diagonal matrix (each block corresponds to one eigenvalue). The reduction of arbitrary saddle-node vector field to this form is made by linear change of variables (to make the linearization matrix to consist of one-dimensional zero block (correspondent to the t- axis) and a matrix B as above), subsequent killing the (nonresonant) monomials $t^l y_i \frac{\partial}{\partial t}$, $l < k+1$, and multiplication of the new field by appropriate nonzero holomorphic function.

Generically, a vector field (11) does not have a central manifold, i.e., a regular separatrix tangent to the t- axis at 0. At the same time, it always has a formal central manifold: there exists a unique formal $n-1$- component vector Taylor series $\hat{q}(t)$ without terms of degrees less than 2 such that the formal change $\tilde{y} = y - \hat{q}(t)$ of the variable y transforms (11) to a (formal) vector field tangent to the t- axis (i.e., having Taylor series that does not contain the terms $t^l \frac{\partial}{\partial y_j}$, which are nonresonant). Generically, this series diverges. At the same time there exists a covering $\bigcup_{j=0}^m S_j$ of a punctured neighborhood of zero in the t- plane by radial sectors such that in each sector S_j there exists a unique $(n-1)$- dimensional holomorphic vector function $q_j(t)$ that is $C^\infty(\overline{S}_j)$, $q_j(0) = 0$, whose graph $y = q_j(t)$ is tangent to the field. All the vector functions q_j have the same asymptotic Taylor series at zero coinciding with $\hat{q}$ (see Theorem 8 below).

A radial sector S satisfies all the statements from the last item, if it satisfies the conditions of the following

Definition. Let (11) be a vector field as in Remark 28, b_i, $i = 1, \dots, n-1$, be the eigenvalues of the matrix B. In this Subsection by *imaginary (real) dividing ray or line* correspondent to an eigenvalue b_i we mean a radial ray or line from the set $\{\mathrm{Re}(\frac{b_i}{t^k}) = 0\}$ (respectively, $\{\mathrm{Im}(\frac{b_i}{t^k}) = 0\}$). A radial sector in the t- plane is said to be good, if for any b_i it contains exactly one correspondent imaginary dividing ray and its closure does not contain additional imaginary dividing rays.

Example 5. In the conditions of the last Definition let $n = 2$, $b_1 \in \mathbb{R}$. In this case the new definitions of imaginary (real) dividing rays and good sectors coincide with those from Section 2.

Theorem 8. *Let (11) be a vector field as in Remark 28, S be a good sector. There exists a unique $(n-1)$- dimensional vector function $q(t)$ holomorphic in a neighborhood of zero in S, continuous at 0 and having bounded derivative, $q(0) = 0$, such that its graph $y = q(t)$ over S is contained in a phase curve of (11). The vector function q is $C^\infty(0)$ and has asymptotic Taylor series at 0. This series does not depend on the choice of the sector and coincides with the formal central manifold series $\hat{q}$ from the item following Remark 28.*

In the case, when $n = 2$, Theorem 8 is reduced to Theorem 1 (Subsection 2.1). Some higher-dimensional particular cases of Theorem 8 were proved in [11]. Its proof in the general case is presented in Subsections 4.2-4.4.

We consider a generic deformation

$$(11)_\varepsilon \qquad \begin{cases} \dot{y} = G(y, t, \varepsilon) \\ \dot{t} = F(y, t, \varepsilon) \end{cases}$$

of a vector field (11) (correspondent to zero value of the deformation parameter ε) that in particular splits the degenerate singularity 0 of the nonperturbed field into $k + 1$ distinct singularities of the perturbed field. The linearization operator of the perturbed field at each singularity has eigenvalue family that tends to zero, as $\varepsilon \to 0$. The singularity of a generically perturbed vector field possesses a unique local separatrix tangent to the eigenline correspondent to this eigenvalue. We show that for a generic deformation $(11)_\varepsilon$ these ("horizontal") separatrices of the perturbed vector field converge to appropriate sectorial separatrices $y = q_j(t)$ of the nonperturbed field from Theorem 8.

Before the statement of this result we firstly recall the Theorem on the existence of separatrix at a nondegenerate singularity of a holomorphic vector field.

Theorem 9. *Let a holomorphic vector field in $\mathbb{C}^n$ have nondegenerate singularity at 0, $b_1, \dots, b_n$ be the eigenvalues of the correspondent linearization operator. Let $\frac{b_n}{b_i} \notin \mathbb{R}$ for any $i \neq n$. Let l_n be the eigenline correspondent to the eigenvalue b_n. Then there exists a unique separatrix of the field tangent to l_n.*

Theorem 9 follows from theorem 4.3.2 in [12] applied to a constant multiple of the vector field under considerations whose linearization operator eigenvalue correspondent to l_n is purely imaginary.

Without loss of generality we consider families $(11)_\varepsilon$ such that all the singularities of the perturbed field lie in the t- axis (their t- coordinates will be referred to, as $\alpha_i(\varepsilon)$, $i = 0, \dots, k$). One can achieve this by applying appropriate continuous family of changes of the variables y (e.g., $\tilde{y} = G(y, t, \varepsilon)$). Define $p(t, \varepsilon) = \prod_{i=0}^{k}(t - \alpha_i(\varepsilon))$. Then $G(0, \alpha_i(\varepsilon), \varepsilon) = 0$, and hence, $G(y, t, \varepsilon) = By + o(|y|) + O(p(t, \varepsilon))$, as $(y, t, \varepsilon) \to 0$. Without loss of generality we also consider that $F(y, t, \varepsilon) = p(t, \varepsilon) + O_2(y, t, \varepsilon)$, $O_2(y, t, \varepsilon) = o(|y|)$, as $(y, t, \varepsilon) \to 0$ (one can achieve this by applying appropriate continuous family of linear t- coordinate changes), and $\sum_{i=0}^{k} \alpha_i(\varepsilon) \equiv 0$.

We prove the statement from the item preceding Theorem 9 for the following families $(11)_\varepsilon$.

Definition. A family $(11)_\varepsilon$ as in the item following Theorem 9 is said to be nondegenerate, if each line passing through a singularity pair of the perturbed field intersects each real dividing line by angle bounded away from zero uniformly in ε, and in the case, when $k \geq 2$, the correspondent polynomial $p(t,\varepsilon)$ satisfies the conditions from the second item of Subsection 2.3.B.

Remark 29. Let $(11)_\varepsilon$ be a nondegenerate family, α_j be its continuous singularity coordinate family. In the case, when $k = 1$, let $V_{j,\varepsilon}$ be the radial sector with the angle π containing $\alpha_j(\varepsilon)$ and symmetric with respect to its radial line (see Fig.5a). The sectors $V_{j,\varepsilon}$ are good, and their boundaries do not accumulate to no imaginary dividing ray. The closure of their union is contained in a good sector. (The latter will be referred to, as S_j.) To the family $\alpha_j(\varepsilon)$ we put into correspondence the sector S_j. In the case, when $k \geq 2$, let A_j be the vertex of the limit regular polygon from Remark 10 correspondent to α_j, V_j be the radial sector with the angle $\frac{\pi}{k}$ containing A_j and symmetric with respect to its radial line. The sector V_j is good. To the family $\alpha_j(\varepsilon)$ we put into correspondence a good sector $S_j \supset V_j$. (The sectors S_j do not cover a neighborhood of zero in the case, when $k \geq 2$, cf. Remark 12.)

Example 6. In the case, when $n = 2$ and the linearization operator of the nonperturbed field $(11)_0$ has real eigenvalues, the above nondegeneracy Definition and the definition of the sectors correspondent to the singularity coordinate families from the previous Remark coincide with those from Subsection 2.3 (with the only difference that in the case, when $k = 1$, the sectors S_j in Remark 29 are chosen to be large enough to contain the closure of the union (in ε) of the sectors $V_{j,\varepsilon}$).

Remark 30. Let $(11)_\varepsilon$ be a nondegenerate family, $\alpha_j(\varepsilon)$ be its continuous singularity coordinate family. Let $\mu(\varepsilon)$ be the eigenvalue family of the correspondent linearization operator that tends to 0, as $\varepsilon \to 0$. For all ε small enough the perturbed field $(11)_\varepsilon$ satisfies the conditions of Theorem 9 at $(0, \alpha_j(\varepsilon))$ with respect to the eigenvalue $b_n = \mu(\varepsilon)$. Moreover, for any other continuous eigenvalue family $b(\varepsilon)$ of the linearization operator the argument $\arg(\frac{\mu(\varepsilon)}{b(\varepsilon)})$ is bounded away from $\pi\mathbb{Z}$ uniformly in all ε small enough. In particular, for any small ε the field $(11)_\varepsilon$ possesses a separatrix at $(0, \alpha_j(\varepsilon))$ (it will be referred to, as $\Gamma_{j,\varepsilon}$) tangent to the eigenline of the linearization operator correspondent to $\mu(\varepsilon)$ (Theorem 9). (This eigenline approaches the line tangent to the t- axis, as $\varepsilon \to 0$.)

Theorem 10. *Let $(11)_\varepsilon$, $\alpha = \alpha_j(\varepsilon)$, $S = S_j$ be as in Remark 29. Let $\Gamma_\varepsilon = \Gamma_{j,\varepsilon}$ be the correspondent separatrix family from the previous Remark. Let q be the vector function from Theorem 8 correspondent to the nonperturbed field and the sector S. There exist an $r > 0$ and a family Ω_ε of domains in the t- plane containing $\alpha(\varepsilon)$ with the following properties:*

1) the connected component of the intersection $\Omega_\varepsilon \cap (S^r \setminus \bigcup_{\alpha_s \neq \alpha} [0, \alpha_s(\varepsilon)])$ containing $\alpha(\varepsilon)$ tends to S^r, as $\varepsilon \to 0$ (see footnote 1).

2) For all $\varepsilon \neq 0$ small enough the separatrix Γ_ε contains the graph $y = q_\varepsilon(t)$ of a vector function $q_\varepsilon(t)$ holomorphic in $t \in \Omega_\varepsilon$ depending on the parameter, $q_\varepsilon(\alpha(\varepsilon)) = 0$, such that

$$\lim_{\varepsilon \to 0} q_\varepsilon|_{\Omega_\varepsilon \cap S^r} = q \quad \textit{(see footnote 2).}$$

In the case, when $n = 2$, Theorem 10 is equivalent to Lemma 2 from Subsection 3.1. In higher dimensions it is proved in Subsections 4.2-4.3.

4.2. Scheme of the proofs of Theorems 8 and 10.

For the proof of Theorems 8 and 10 we show (in the next Subsection) that in the conditions of Theorem 10 the separatrices Γ_ε contain graphs of vector-functions $q_\varepsilon(t)$ holomorphic in domains Ω_ε satisfying statement 1) of Theorem 10 such that inequalities (3) from Subsection 3.2 hold. (In particular, the family q_ε is normal in the sector S^r.) Then each limit q of convergent sequence q_{ε_m}, $\varepsilon_m \to 0$, satisfies the statements of the first part of Theorem 8: it is continuous in $\overline{S^r}$, vanishes at 0 and has bounded derivative; its graph over S^r is tangent to the nonperturbed field. For the proof of Theorem 8 we consider a nondegenerate deformation of the given vector field (11) such that some its continuous singularity family corresponds to the sector S. We show (in Subsection 4.4) that any vector function q satisfying the three previous statements is $C^\infty(0)$ and has the asymptotic Taylor series $\hat{q}$ at 0. We prove uniqueness of such a vector function at the end of the paper. This will prove Theorem 8.

The convergence statement of Theorem 10 follows from the statements on the vector-functions q_ε from the beginning of the previous item (inequalities (3) and statement 1) of Theorem 10) and the uniqueness statement of Theorem 8, as in the proof of Lemma 2.

4.3. Proof of Theorem 10.

In higher dimensions a version of Theorem 10 was implicitly proved in [6] under the assumption that $(11)_\varepsilon$ is tangent to the spaces $t = \alpha_i(\varepsilon)$ for all ε and $i = 0, \dots, k$ (i.e., in the notations of the item following Theorem 9 without loss of generality one can consider that $F \equiv p$ $(O_2 \equiv 0)$). Though a proof in [6] was presented for vector field families obtained by projectivization of linear differential equations, it remains valid in this more general case. In difference with the two-dimensional case, in general, in higher dimensions a vector field (11) does not necessarily have an integral hypersurface tangent to the y- space at 0, and in this case no its deformation $(11)_\varepsilon$ is orbitally analytically equivalent to a one with $F \equiv p$. Below we modify the proof from [6] in order to make it valid for the general case.

Without loss of generality we consider that the t- axis is tangent to the separatrix G_ε of the field $(11)_\varepsilon$ at the singular point $(0, \alpha(\varepsilon))$. One can achieve this by applying appropriate continuous family of changes of the coordinates y. Then the separatrix Γ_ε contains the graph $y = q_\varepsilon(t)$ of a vector function q_ε holomorphic in a neighborhood of $\alpha(\varepsilon)$ depending on ε, $q_\varepsilon(\alpha(\varepsilon)) = q'_\varepsilon(\alpha(\varepsilon)) = 0$. For the proof of Theorem 10 it suffices to show that the vector functions q_ε are holomorphic in appropriate domains Ω_ε (satisfying statement 1) of Theorem 10) and satisfy inequalities (3) from Subsection 3.2. Let $K = (|\dot{t}| > |\dot{y}|)$ and $\widetilde{K} = \{|t - \alpha(\varepsilon)| > |y|\}$ be respectively the tangent cone field and the correspondent cone from the same Subsection. Inequalities (3) are equivalent to inclusions (4) from the same Subsection in the graphs $y = q_\varepsilon(t)$. They hold a priori in a neighborhood of $\alpha(\varepsilon)$ depending on ε. To show that they hold in a large domain, we consider a pair of vector field families $(11)_{1,\varepsilon} = e^{i\theta_1}(11)_\varepsilon$, $(11)_{2,\varepsilon} = e^{i\theta_2}(11)_\varepsilon$, $\theta_j \in \mathbb{R}$, which are constant multiples of $(11)_\varepsilon$, that possess the following properties:

1) The eigenvalue families of the linearization operators of the vector fields $(11)_{1,\varepsilon}$, $(11)_{2,\varepsilon}$ at the singular point $(0, \alpha(\varepsilon))$ correspondent to the eigenline tangent to the t- axis have positive real parts, and their arguments are bounded away from $\pi\mathbb{Z}$ uniformly in ε small enough.

2) The variables y_i are split into two groups: the "stable" $y_s = (y_{s_1}, \dots, y_{s_l})$ and

the "unstable" $y_u = (y_{u_1}, \dots, y_{u_q})$ (one of them may be empty). The tangent spaces of the y_s- and y_u- subspaces at $(0,0)$ are invariant with respect to the linearization operator of the nonperturbed field $(11)_0$ (and hence, to that of each of the nonperturbed fields $(11)_{1,0}$ and $(11)_{2,0}$ from the new families). The eigenvalues of the linearization operator of the field $(11)_{1,0}$ correspondent to the y_s- subspace have negative real parts; those correspondent to the y_u- subspace have positive real parts. Those of the linearization operator of the field $(11)_{2,0}$ correspondent to the y_s- (y_u-) subspace have oppositely positive (respectively, negative) real parts.

For a pair of real numbers θ_1 and θ_2 let us define the vector field families $w_j(\varepsilon)$: $\dot{t} = e^{i\theta_j}p(t,\varepsilon)$ in the t- plane, $j = 1, 2$, which are the projection images to the latter of the restrictions to it of the vector field families $(11)_{j,\varepsilon}$. The possibility of choice of θ_j satisfying the conditions 1) and 2) is implied by the following

Remark 31. In the conditions of Theorem 10 let $SI \subset S$ be the closure in S of the maximal subsector bounded by imaginary dividing rays. In the case, when $k = 1$, let $V_\varepsilon = V_{j,\varepsilon}$ be the family of good sectors from Remark 29 correspondent to $\alpha(\varepsilon)$, V be a good sector contained in the interior of the intersection $\cap_\varepsilon V_\varepsilon$. In the case, when $k \geq 2$, let $V = V_j$ be the correspondent sector from Remark 29. Let θ_1, θ_2 be chosen so that the correspondent nonperturbed vector fields $w_1(0)$, $w_2(0)$ of the families from the previous item have repelling rays lying in V on different sides from SI: that correspondent to the former lies on the right and that correspondent to the latter lies on the left. Then θ_1 and θ_2 satisfy the previous conditions 1) and 2) for appropriate coordinates (y_s, y_u). The proof of this statement is implicitly contained in remark 24 of [6]. One can choose θ_i as at the beginning of the item so that $\theta_1 > \theta_2$ and $\theta_1 - \theta_2 < \pi$. This follows from the fact that the angle of the sector V is not greater than $\frac{\pi}{k}$ by definition.

Remark 32. In the conditions of Theorem 10 let θ_1, θ_2, $(11)_{1,\varepsilon}$, $(11)_{2,\varepsilon}$, (y_s, y_u) satisfy conditions 1) and 2) preceding Remark 31, $w_l(\varepsilon)$, $l = 1, 2$, be the correspondent vector field families in the t- line from the item preceding the previous Remark. Then $(0, \alpha(\varepsilon))$ is a hyperbolic singular point for both $(11)_{1,\varepsilon}$ and $(11)_{2,\varepsilon}$. The correspondent unstable manifold of the field $(11)_{1,\varepsilon}$ has tangent space at $(0, \alpha(\varepsilon))$ converging to that of the coordinate (y_u, t) -space, as $\varepsilon \to 0$. The tangent space at $(0, \alpha(\varepsilon))$ of the unstable manifold of the field $(11)_{2,\varepsilon}$ tends to that of the (y_s, t)-space. In particular, for small ε the former intersects the latter transversally at $(0, \alpha(\varepsilon))$. The intersection of the unstable manifolds contains Γ_ε and coincides with it locally in a neighborhood of $(0, \alpha(\varepsilon))$. The conjoint singular point $\alpha(\varepsilon)$ of both perturbed fields $w_l(\varepsilon)$, $l = 1, 2$, is repelling, and the arguments of the correspondent multipliers are bounded away from $\pi\mathbb{Z}$ uniformly in ε.

Let θ_1 and θ_2 be as in Remark 31. Recall that by assumption, the linearization matrices of the fields $(11)_{1,0}$ and $(11)_{2,0}$ at 0 are upper-triangular and block-diagonal (the eigenvalues of each block coincide; in particular, each block corresponds to either the t- axis, or a subspace of the coordinate y_s- (y_u-) space). Without loss of generality we consider that their nondiagonal terms are small enough with respect to the nonzero eigenvalue real parts so that the sum of the moduli of the formers multiplied by 3 is less than the minimum of the moduli of the real parts of the nonzero eigenvalues. For the proof of inclusions (4) we consider the two following

tangent cone fields:

$$K_1 = (|\dot t| + |\dot y_u| > 3\,|\dot y_s|) \text{ and } K_2 = (|\dot t| + |\dot y_s| > 3|\dot y_u|),$$

where

$$|\dot y_u| = \sum_i |\dot y_{u_i}|, \ \ |\dot y_s| = \sum_j |\dot y_{s_j}|.$$

We also consider the correspondent cones

$$\widetilde K_1 = \{|t - \alpha(\varepsilon)| + |y_u| > 3\,|y_s|\}, \ \widetilde K_2 = \{|t - \alpha(\varepsilon)| + |y_s| > 3\,|y_u|\}.$$

We show that

$$T\,\Gamma_\varepsilon \subset K_1, \ T\,\Gamma_\varepsilon \subset K_2, \ \Gamma_\varepsilon \subset \widetilde K_1 \cap \widetilde K_2 \tag{12}$$

over appropriate set Ω_ε. Then (12) will imply (4), since $K \supset (K_1 \cap K_2)$ and $\widetilde K \supset \widetilde K_1 \cap \widetilde K_2$.

The fields $(11)_{1,\varepsilon}$ and $(11)_{2,\varepsilon}$ are tangent to Γ_ε. Their restrictions to the latter have repelling singularity at $(0, \alpha(\varepsilon))$. For all ε small enough inclusions (12) hold locally in a neighborhood of the singularity (that depends on ε). To show that (12) hold in a large domain, we use the fact that the cone field K_1 (K_2) is invariant with respect to the vector field $(11)_{1,\varepsilon}$ ($(11)_{2,\varepsilon}$) in a neighborhood U of zero in the phase space independent on ε for all ε small enough. The proof of this statement uses only conditions 1), 2) preceding Remark 31 and the assumption from the item preceding the definition of the cone field K_1 and is analogous to that of proposition 3 in [6].

Without loss of generality we consider that $U = U_y \times U_t$, $U_y = \{|y_s| < 2\delta\} \times \{|y_u| < 2\delta\}$, $U_t = \{|t| < \delta\}$. Let $L\Omega^1_\varepsilon$ and $L\Omega^2_\varepsilon$ be the subdomains of $\Gamma_\varepsilon \cap U$ saturated by the trajectories of the field $(11)_{1,\varepsilon}$ (respectively, $(11)_{2,\varepsilon}$) in $\Gamma_\varepsilon \cap U$ that go from $(0, \alpha(\varepsilon))$. Let $L\Omega_\varepsilon$ be the connected component of the intersection $L\Omega^1_\varepsilon \cap L\Omega^2_\varepsilon$ that contains $(0, \alpha(\varepsilon))$. Inclusions (12) hold in $L\Omega_\varepsilon$ by invariance of the cone fields and the fact that they hold locally in a neighborhood of the singularity. For the proof of Theorem 10 it suffices to show that (12) hold in the graphs $y = q_\varepsilon(t)$ of vector functions q_ε holomorphic in appropriate domains Ω_ε satisfying statement 1) of Theorem 10. To do this, it suffices to show that $L\Omega_\varepsilon$ contain such graphs. We prove this for domain families Ω_ε of the following type.

Definition of the domains Ω_ε. Let $U_t = \{|t| < \delta\}$ be a neighborhood of zero in the t- plane, θ_i, $w_i(\varepsilon)$, $i = 1, 2$, be as in Remark 31. Let Ω^i_ε be the family of domains saturated by the trajectories of the field $w_i(\varepsilon)$ in U_t that go from $\alpha(\varepsilon)$. Define Ω_ε to be the connected component of the intersection $\Omega^1_\varepsilon \cap \Omega^2_\varepsilon$ that contains $\alpha(\varepsilon)$ (see Fig.4a in the case, when $k = 1$).

There exist θ_1 and θ_2 as in Remark 31 such that the correspondent family Ω_ε from the previous Definition satisfies statement 1) of Theorem 10. Namely, this is the case, provided that the correspondent repelling rays from Remark 31 are close enough to the imaginary dividing rays that form the boundary of the sector SI from the latter, as in the proof of the analogous statement in Subsection 3.2.

For the proof of Theorem 10 it would be sufficient to show that $L\Omega_\varepsilon$ contains the graph of a vector function $q_\varepsilon(t)$ holomorphic in Ω_ε. In the case, when $O_2 \equiv 0$,

$L\Omega_\varepsilon$ is such a graph itself: it is 1-to-1 projected onto Ω_ε (see subsection 5.B in [6]). This is the place in the proof of the particular case of Theorem 10 from [6] we used the last equality.

Let Ω_ε be a given domain family as in the previous Definition. In the case, when $O_2 \not\equiv 0$, for the proof of Theorem 10 we consider another pair of numbers θ_i', $i = 1, 2$, satisfying the conditions of Remark 31 such that $\theta_1 > \theta_1' > \theta_2' > \theta_2$: the ordering of the "arguments" $\theta_2, \theta_2', \theta_1', \theta_1$ defines the counterclockwise order of the correspondent radial rays. We consider the correspondent domain family $L\Omega_\varepsilon'$ from the item preceding the previous Definition. We show that the new domain $L\Omega_\varepsilon'$ (where inclusions (12) hold by the discussion from the same place) contains the graph of a vector function $q_\varepsilon(t)$ holomorphic in Ω_ε, whenever δ and ε are small enough. This will prove Theorem 10.

The projection of the new domain $L\Omega_\varepsilon'$ to the t- plane is a local diffeomorphism. This follows from the fact that the tangent lines to $L\Omega_\varepsilon'$ are contained in the tangent cone field $K \supset K_1 \cap K_2$. Thus, the inverse map $\Omega_\varepsilon \to L\Omega_\varepsilon'$ is well-defined a priori in a neighborhood of the point $\alpha(\varepsilon)$ and extends analytically to any its simply connected neighborhood disjoint from the projection image of $\partial L\Omega_\varepsilon'$. We show that the whole domain Ω_ε is disjoint from the latter, whenever δ and ε are small enough. This will imply the last statement from the last item. To do this, we use the following properties of $L\Omega_\varepsilon'$.

Remark 33. Let θ_1, θ_2 be as in Remark 31, U, U_y, U_t, $L\Omega_\varepsilon$ be as in the two items preceding the previous Definition. The domain $L\Omega_\varepsilon$ meets the boundary of the neighborhood U at points of the cylinder $U_y \times \partial U_t$. The domain $L\Omega_\varepsilon$ is invariant with respect to the vector fields $-(11)_{1,\varepsilon}$ and $-(11)_{2,\varepsilon}$, i.e., for any $a \in L\Omega_\varepsilon$ the arcs of their trajectories that connect a to the singular point $(0, \alpha(\varepsilon))$ are contained in $L\Omega_\varepsilon$. The first statement of the item follows from the inclusion $L\Omega_\varepsilon \subset \widetilde{K}$ ($TL\Omega_\varepsilon \subset K$) in the same way, as in the proof of lemma 1 in subsection 5.B of [6]. Let us prove the second statement of the item, e.g., for the first vector field. Let $L\Omega_\varepsilon^1$, $L\Omega_\varepsilon^2$ be as in the item preceding the previous Definition. The domain $L\Omega_\varepsilon^1$ ($L\Omega_\varepsilon^2$) is $-(11)_{1,\varepsilon}$- (respectively, $-(11)_{2,\varepsilon}$-) invariant. Therefore, by the latter and definition, the boundary of the domain $L\Omega_\varepsilon$ consists of parts of $\partial L\Omega_\varepsilon^1$ and some (may be, semiinfinite) arcs of trajectories of the field $(11)_{2,\varepsilon}$ that start and end at $\partial L\Omega_\varepsilon^1$. Each of these arcs splits the domain $L\Omega_\varepsilon^1$ into two connected components. Let L be one of these arcs. By $L\Omega_\varepsilon^0$ denote the correspondent splitting component of $L\Omega_\varepsilon^1$ that contains the singular point. It suffices to show that the field $(11)_{1,\varepsilon}$ is directed outside $L\Omega_\varepsilon^0$ at L. This follows from the fact that the angle between the vector fields is constant (in particular, $(11)_{1,\varepsilon}$ has constant angle with L) and there exists a point of L where $(11)_{1,\varepsilon}$ is directed outside $L\Omega_\varepsilon^0$. The last statement is implied by the definition of the domain $L\Omega_\varepsilon^1$: each point of the complement $L\Omega_\varepsilon^1 \setminus \text{Ł}\Omega_\varepsilon^0$ is connected to the singular point $(0, \alpha(\varepsilon))$ by trajectory of the field $(11)_{1,\varepsilon}$ in $L\Omega_\varepsilon^1$.

Corollary 6. *In the conditions of the previous Remark the boundary $\partial L\Omega_\varepsilon$ consists of the two following parts:*

1) arcs in the cylinder $U_y \times \partial U_t$ where both vector fields $(11)_{1,\varepsilon}$ and $(11)_{2,\varepsilon}$ are directed outside the latter;

2) arcs of semitrajectories of the fields $(11)_{1,\varepsilon}$ and $(11)_{2,\varepsilon}$ in $\overline{U}$ that start at some points of their tangency with the cylinder.

Example 7. Let $k=1$. Then in the conditions of the previous Remark one can show that for U and ε small enough the part 1) of the boundary $\partial L\Omega_\varepsilon$ consists of a single arc and the part 2) consists generally of two positive semitrajectories of the fields $(11)_{1,\varepsilon}$ and $(11)_{2,\varepsilon}$ that converge to the other singular point $(0,-\alpha(\varepsilon))$. (In some exceptional cases it consists of two finite arcs of these semitrajectories that have a common end.) The projection images of these semitrajectories in the t- plane are close to the semitrajectories of the correspondent vector fields $w_1(\varepsilon)$ and $w_2(\varepsilon)$ that form the boundary of Ω_ε (see Fig. 4a and Proposition 6 below). The semitrajectories of the latters intersect each other, so, the correspondent part of the boundary of the domain Ω_ε consists of their arcs having a common end. In particular, the projection images of the previous semitrajectories (forming $\partial L\Omega_\varepsilon$) intersect each other. At the same time, one can show that generically these semitrajectories do not intersect each other (so, the projection images in the t- plane of some disjoint parts of $L\Omega'_\varepsilon$ overlap). The proof of this statement is omitted to save the space.

Now let us show that the projection image in the t- plane of the boundary $\partial L\Omega'_\varepsilon$ does not intersect Ω_ε, whenever U and ε are small enough. The projection image of the part 1) of the boundary from Corollary 6 does not intersect Ω_ε by definition. Thus, it suffices to show that no projection of arc from the part 2) (which starts outside Ω_ε) can enter Ω_ε. To do this, we use the following

Proposition 5. *Let $p(t,\varepsilon)=\prod_{i=0}^{k}(t-\alpha_i(\varepsilon))$ be a monic polynomial vector field family that satisfies the asymptotic singularity polygon regularity statement from Remark 10 in the case, when $k\geq 2$. Let $(\theta_1,\ \theta_2)$, (θ'_1,θ'_2), be real number pairs such that $\theta_1>\theta'_1>\theta'_2>\theta_2$, $\theta_1-\theta_2<\pi$ (i.e., the ordering of the "arguments" $\theta_2,\theta'_2,\theta'_1,\theta_1$ defines the counterclockwise order of the correspondent radial rays in complex plane). Let $(w_1(\varepsilon),\ w_2(\varepsilon))$, $(w'_1(\varepsilon),\ w'_2(\varepsilon))$ be the correspondent vector field family pairs from the item preceding Remark 31. Let $\alpha(\varepsilon)=\alpha_i(\varepsilon)$ be their conjoint singular point family such that the correspondent multipliers of the four perturbed fields have positive real parts and their arguments are bounded away from $\pi\mathbb{Z}$. Let U_t be a neighborhood of zero in the t- line, Ω_ε be the domain family from the previous Definition correspondent to the first vector field family pair. The domain Ω_ε is invariant with respect to the fields $-w'_1(\varepsilon)$ and $-w'_2(\varepsilon)$. Each of the latters is directed strictly inside Ω_ε, and moreover its angle with the arcs of the boundary $\partial\Omega_\varepsilon$ is bounded away from $\pi\mathbb{Z}$ uniformly in U_t and ε.*

Proof. The domain Ω_ε is $-w_1(\varepsilon)$- and $-w_2(\varepsilon)$- invariant, as is $L\Omega_\varepsilon$ with respect to the correspondent vector fields in Remark 33. Let $t\in\partial\Omega_\varepsilon$. Let us show that the fields $w'_1(\varepsilon)$ and $w'_2(\varepsilon)$ are directed outside Ω_ε at t. Let $W=W(t,\varepsilon)$ be the sector in the tangent plane at t with angle less than π formed by the vectors of the fields $w_1(\varepsilon)$ and $w_2(\varepsilon)$. Each radial vector in W is directed outside Ω_ε. This follows from the statement that both $w_1(\varepsilon)$ and $w_2(\varepsilon)$ are not directed inside Ω_ε in its boundary (the previous invariance statement). The vectors of the fields $w'_1(\varepsilon)$ and $w'_2(\varepsilon)$ are contained in W and have constant angles with its boundary rays by definition. This proves Proposition 5.

Corollary 7. *In the conditions of Proposition 5 the domain Ω'_ε from the previous Definition correspondent to the fields $w'_i(\varepsilon)$, $i=1,2$, contains Ω_ε.*

Let $(11)'_{1,\varepsilon}$, $(11)'_{2,\varepsilon}$ be the vector fields from the beginning of the proof of Theorem 10 correspondent to the new numbers θ'_1 and θ'_2, $L\Omega'_\varepsilon$ be the correspondent domain

from the item preceding the previous Definition. The arcs from the part 2) of $\partial L\Omega'_\varepsilon$ lie in trajectories of the fields $(11)'_{1,\varepsilon}$ and $(11)'_{2,\varepsilon}$. Now the last statement preceding Proposition 5 will follow from the latter and the fact that the projection images in the t- plane of the vectors of these fields restricted to the closure $\overline{L\Omega'_\varepsilon}$ have arbitrarily small angles with the field $w'_1(\varepsilon)$ (respectively, $w'_2(\varepsilon)$), provided that U and ε are small enough. Then these angles can be made so small that these projection vectors could not be directed inside Ω_ε, as the fields $w'_i(\varepsilon)$ in Proposition 5. Thus, no projection image of arc from part 2) of the boundary $\partial L\Omega'_\varepsilon$, which starts outside Ω_ε, can enter the latter.

The projection to the t- line of the closure $\overline{L\Omega'_\varepsilon}$ is local diffeomorphism outside the singular points of the field $(11)_\varepsilon$: the restriction of $(11)_\varepsilon$ to $\overline{L\Omega'_\varepsilon}$ lies in the cone K closure field, since so does the restriction to $L\Omega'_\varepsilon$. The angles from the last item are equal to the angle between the vector field $(11)_\varepsilon$ and the lifting to $\overline{L\Omega'_\varepsilon}$ of the field $p(t,\varepsilon)$. The angle bound statement from the last item is implied by the following

Proposition 6. *Let $(11)_\varepsilon$, α be as in Theorem 10. Let K and $\widetilde{K}$ be the cone field and the cone from the beginning of the Subsection. For any ε (including $\varepsilon = 0$) let $K_\varepsilon \subset \overline{\widetilde{K}} \setminus \{(0, \alpha_i(\varepsilon)); i = 0, \dots, k\}$ be the subset of points where the vectors of the field $(11)_\varepsilon$ are contained in the cone closure field $\overline{K}$. Let $p(t,\varepsilon)$ be the polynomial family from the item following Theorem 9, $(11)''_\varepsilon$ be the vector field in K_ε contained in the complex tangent line field correspondent to $(11)_\varepsilon$ and projected to the vector field $\dot{t} = p(t,\varepsilon)$ in the t- plane. Then for any $\sigma > 0$ there exists a neighborhood U of zero in the phase space such that for all ε small enough the inequality $|(11)_\varepsilon - (11)''_\varepsilon| < \sigma |(11)''_\varepsilon|$ holds in $K_\varepsilon \cap U$.*

Proposition 6 is proved below.

In the conditions of the item following Corollary 7 for any $\sigma > 0$ and any U, ε small enough (dependently on σ) the angle between the field $(11)'_{l,\varepsilon}|_{L\Omega'_\varepsilon}$ and the lifting to $L\Omega'_\varepsilon$ of the field $w'_i(\varepsilon)$, $i = 1, 2$, is less than σ everywhere in $\overline{L\Omega'_\varepsilon}$. This follows from Proposition 6 and the inclusion $L\Omega'_\varepsilon \subset K_\varepsilon$, which holds whenever U and ε are small enough. This together with Proposition 5 and Corollary 6 proves that for U and ε small enough the domain Ω_ε is disjoint from the projection image of the boundary of $L\Omega'_\varepsilon$. Theorem 10 is proved modulo Proposition 6.

Proof of Proposition 6. Let B be the matrix from expression (11) for the nonperturbed vector field, $G(y,t,\varepsilon)$, $F(y,t,\varepsilon)$, $p(t,\varepsilon)$ be the (vector) functions from the definition of $(11)_\varepsilon$ and the item following Theorem 9. For the proof of Proposition 6 it suffices to show that $F(y,t,\varepsilon) = p(t,\varepsilon)(1 + o(1))$, as (y,t,ε) tends to 0 so that $(y,t) \in K_\varepsilon$. Suppose the contrary, i.e., there exists a $c > 0$ such that the inequality $|F - p| > c|p|$ holds along a sequence $(y_m, t_m, \varepsilon_m) \to 0$, $(y_m, t_m) \in K_{\varepsilon_m}$. By the assumptions from the item following Theorem 9, $F(y,t,\varepsilon) = p(t,\varepsilon) + o(|y|)$. Therefore, both p and F are $o(|y|)$ along this sequence. Hence, so is G, which follows from the same statement for F and the inequality $|G| \leq |F|$ (the definition of the set K_ε in Proposition 6). On the other hand, $G(y,t,\varepsilon) = By + o(|y|)$ along the same sequence, which follows immediately from the expression for G in the item following Theorem 9 and the asymptotic $p(t,\varepsilon) = o(|y|)$ along the sequence proved before. Therefore, $G(y,t,\varepsilon) \neq o(|y|)$ along the sequence (the matrix B is nondegenerate). This contradicts the previous statement that $G(y,t,\varepsilon) = o(|y|)$ along the

sequence. The obtained contradiction proves Proposition 6. The proof of Theorem 10 is completed.

4.4. Proof of Theorem 8.

Let (11), S be as in Theorem 8. For the proof of Theorem 8 we consider a nondegenerate deformation $(11)_\varepsilon$ of the field (11) with the following properties.

Remark 34. For any vector field (11) and sector S as in Theorem 8 there exists a nondegenerate deformation $(11)_\varepsilon$ of (11) (satisfying the assumptions of the item following Theorem 9) with a singularity coordinate family $\alpha(\varepsilon)$ such that the correspondent sector from Remark 29 contains the same imaginary dividing rays, as S. Indeed, the right-hand side in (11) is $O(|y|) + t^{k+1}g(t)$ where g is a holomorphic vector function. Changing the multiplier t^{k+1} in the last formula to any given degree $k+1$ monic polynomial family $p(t,\varepsilon)$, $p(t,0) = t^{k+1}$, yields a deformation $(11)_\varepsilon$ satisfying the assumptions from the item following Theorem 9. Let $\theta \in \mathbb{R}$ be a fixed constant. Let us choose a polynomial family $p(t,\varepsilon)$ dependent on the real parameter ε as follows: $p(t,\varepsilon) = t^{k+1} - e^{i\theta}\varepsilon = \prod_{j=0}^{k}(t - e^{i(\theta+\frac{2\pi}{k+1}j)}\varepsilon^{\frac{1}{k+1}})$, so, the correspondent root families $\alpha_j(\varepsilon) = e^{i(\theta+\frac{2\pi}{k+1}j)}(\varepsilon)^{\frac{1}{k+1}}$ form a regular polygon family centered at 0 with constant vertex radial rays. Let $SI \subset S$ be the maximal subsector bounded by imaginary dividing rays. A deformation $(11)_\varepsilon$ defined by a polynomial family $p(t,\varepsilon)$ as above will be a one we are looking for, provided that θ is chosen so that the radial ray of some root family $\alpha = \alpha_j(\varepsilon)$ is close enough to the bissectrix of the sector SI and no root radial ray coincides with a real dividing ray.

Let Γ_ε be the separatrices of the perturbed fields $(11)_\varepsilon$ at $(0,\alpha(\varepsilon))$ from Remark 30. The separatrices Γ_ε contain the graphs $y = q_\varepsilon(t)$ of holomorphic vector functions $q_\varepsilon(t)$ defined in domains Ω_ε satisfying statement 1) of Theorem 10 such that $|q'_\varepsilon| < 1$, $|q_\varepsilon(t)| < |t - \alpha(\varepsilon)|$ (for $t \neq \alpha(\varepsilon)$). These statements were proved in the previous Subsection. Then the vector function family q_ε is normal, so any sequence q_{ε_m}, $\varepsilon_m \to 0$, contains a subsequence convergent uniformly in compact subsets in S^r. Each limit $q(t)$ of convergent sequence q_{ε_m}, $\varepsilon_m \to 0$, as $m \to \infty$, is a vector function $q(t)$ holomorphic in the sector S^r and continuous in its closure such that a) $q(0) = 0$; b) $|q'|_{S^r} \leq 1$; c) the graph $y = q(t)$ over the sector S^r is contained in a phase curve of the nonperturbed field. (Statements a) and b) follow from the previous inequalities.) Let us show that a vector function $q(t)$ with properties a), b), c) satisfies the statements of Theorem 8.

Firstly let us prove that q has asymptotic Taylor series at 0 (coinciding with the formal central manifold series $\hat{q}$ from the beginning of the Section). To do this, let us prove the correspondent asymptotic Taylor formula. Namely, for $m \in \mathbb{N}$ define $q_m(t)$ to be the Taylor polynomial of degree at most m coinciding with the m- th partial sum of the series $\hat{q}$. Let us show that $q(t) - q_m(t) = o(t^m)$, as $t \to 0$ for any m. Without loss of generality we consider that $q_m \equiv 0$. One can achieve this by applying the change $y \mapsto y - q_m(t)$ of the coordinates y. Then the vector field (11) takes the form

$$\left\{ \begin{array}{l} \dot{y} = G(y,t) = By + O(|y|^2 + |y||t|) + O(t^{m+1}) \\ \dot{t} = F(y,t) = t^{k+1} + O(|y|^2). \end{array} \right. \tag{13}$$

Let us show that $q(t) = o(t^m)$, as $t \to 0$. In the proof of this statement we use the asymptotic formula

$$F(y,t)|_{y=q(t)} = t^{k+1}(1 + o(1)), \text{as } t \to 0. \tag{14}$$

(This is equivalent to the statement that the angle between the field (11) and the lifting to the graph $y = q(t)$ of the field $\dot{t} = t^{k+1}$ in the t- line tends to 0, as $t \to 0$, cf. the proof of Proposition 6). This follows from inequality b) in the last item (which implies that the graph $y = q(t)$ is contained in the closure of the cone $\widetilde{K}$ from the beginning of the previous Subsection) and Proposition 6 applied to the nonperturbed field and its singular point 0. Consider the coordinates (y_s, y_u) and the vector fields $(11)_{1,0}$, $(11)_{2,0}$, $w_1(0)$, $w_2(0)$ from Remark 31 chosen so that the two latters have repelling rays in S. (In general, the sector S from Theorem 8 does not contain the whole sector V from Remark 31. A priori, it contains the closure of the sector SI from the latter. One can choose the two last fields as in the same Remark so that the correspondent repelling rays will be close enough to the boundary of the sector SI so that they fit the sector S.) The angle between $(11)_{i,0}$ and the lifting to the graph $y = q(t)$ of the field $w_i(0)$ tends to 0, as $t \to 0$. Then the trajectories of the field $-(11)_{i,0}$ that start in the graph over the sector S^{r_0} with appropriate $r_0 > 0$ converge to 0 so that their projections to the t- plane do not leave the sector S^r and converge to zero with the asymptotic tangency to the repelling ray of the field $w_i(0)$. Recall that the linearization operators of the fields $-(11)_{1,0}$, $-(11)_{2,0}$ have invariant subspaces tangent to the y_s- and y_u- spaces; the eigenvalues of that of the field $-(11)_{1,0}$ in the first subspace have positive real parts, and those in the second subspace have negative real parts; those correspondent to the second field have oppositely negative (respectively, positive) real parts. The nondiagonal terms of the linearization matrices of these fields are small with respect to the real parts of nonzero eigenvalues, more precisely, satisfy the assumptions from the item following Remark 32.

We prove the asymptotic formula $q(t) = o(t^m)$ by contradiction. Suppose the contrary. This means that there exists a $c > 0$ such that there exists arbitrarily small $t_0 \in S^r$ where $|q(t_0)| > c|t_0|^m$. We consider that $|q_s(t_0)| \geq |q_u(t_0)|$, so $|q_s(t_0)| > \frac{c}{2}|t_0|^m$ (the opposite case is analyzed analogously). Let us show that the norm $|q_s(t)|$ increases along the trajectory of the field $-(11)_{1,0}$ that starts at $(q(t_0), t_0)$, provided that t_0 is close enough to 0. Then this will contradict the convergence of this trajectory to 0 and will prove the asymptotic Taylor formula. For the proof of the increasing of this norm we consider the subdomain $K_m = \{|y_s| > |y_u|, \ |y_s| > \frac{c}{2}|t|^m\} \cap \{y = q(t)\}$ of the graph of the vector function $q(t)$. The closure of the domain K_m contains the point $(q(t_0), t_0)$, by definition. We use the fact that K_m is $-(11)_{1,0}$- invariant over a neighborhood of 0 small enough in S^r (more precisely, for any point of ∂K_m close enough to 0 the correspondent positive semitrajectory of the field $-(11)_{1,0}$ enters K_m locally). This statement follows from the definition of the coordinates (y_s, y_u), (13), (14) and is proved by a straightforward calculation of the derivative of the ratios $\frac{|y_s|}{|y_u|}$, $\frac{|y_s|}{|t|^m}$, along the field $-(11)_{1,0}$ in ∂K_m: these derivatives will be positive, whenever $(y = q(t), t) \in \partial K_m$ is close enough to 0, as in the proof of proposition 3 from [6]. For the proof of the previous $|q_s|$ increasing statement it suffices to show that the norm $|y_s|$ increases along the trajectories of the field $-(11)_{1,0}$ in $\overline{K_m}$ at all the points of the latter close enough to 0. Indeed, the increment of the norm $|y_s|$ along a small trajectory segment consists of the following parts: the contribution of the linear terms By in the right-hand side of (13) and that of the nonlinear terms. The former is positive, and moreover it is greater than $|y_s|$ times some positive constant independent on $(y, t) \in K_m$ (by definition of the coordinate subspaces y_s and y_u). It dominates

the nonlinear term contribution, whenever $(y,t) \in \overline{K_m}$ is close enough to 0. This follows from the fact that the nonlinear terms are $o(|y|) + o(|t|^{m+1}) = o(|y_s|)$, as $(y,t) \to 0$ so that $(y,t) \in K_m$ (the definition of K_m). This proves the asymptotic Taylor formula for $q(t)$. The statement of Theorem 8 that $\hat{q}$ is the asymptotic Taylor series of the function q is proved modulo continuity of the derivatives of q at 0. In the proof of the last statement we use the asymptotic Taylor formula proved before and the differential equation for the vector function q:

$$\frac{dq}{dt} = \frac{G(y,t)}{F(y,t)}|_{y=q(t)}.$$

Let us prove continuity of q'. For higher derivatives the proof is analogous. Let us show that $q'(t) \to 0$, as $t \to 0$. Let $q_{k+1}(t)$ be the Taylor polynomial from the previous asymptotic Taylor formula. Without loss of generality we consider that $q_{k+1} \equiv 0$, as in the item preceding (13). Then $q(t) = o(t^{k+1})$ by the asymptotic Taylor formula, hence $G(q(t),t) = Bq(t) + O(|q(t)|^2 + |tq(t)| + |t|^{k+2}) = o(t^{k+1})$. This together with the previous differential equation and (14) implies that $q'(t) = o(1)$, as $t \to 0$. This proves the continuity of the extension of q' to 0 by zero.

Now let us prove the uniqueness of a vector function q with properties a) and c) from the beginning of the proof of Theorem 8 and bounded derivative. Suppose there exists another such vector function $\tilde{q} \neq q$. Without loss of generality we consider that the module of its derivative is not greater than 1 (i.e., inequality b) from the beginning of the Subsection holds). One can achieve this by applying appropriate linear change $y \mapsto \lambda y$ of the coordinates y. Then by the previous discussion, $\tilde{q}$ has asymptotic Taylor series at 0. We consider that $q \equiv 0$. One can achieve this by applying the change $y \mapsto y - q(t)$ of the variable y over the sector S^r. (Then the new vector field (11) will be holomorphic over S^r and in general, only C^∞ at 0 with asymptotic Taylor series.) Let us show that $\tilde{q} \equiv 0$.

Suppose the contrary: there exists a $t_0 \in S^r$ where $\tilde{q}(t_0) \neq 0$. We consider that $|\tilde{q}_s(t_0)| \geq |\tilde{q}_u(t_0)|$ (the opposite case is analyzed analogously). The semitrajectory of the field $-(11)_{1,0}$ that starts at $(\tilde{q}(t_0), t_0)$ converges to 0 so that its projection to the t- line does not leave the sector S^r, provided that t_0 is small enough, as in the proof of the asymptotic Taylor formula. Let us show that the norm $|\tilde{q}_s(t_0)|$ increases along this trajectory, if t_0 is small enough. This will contradict its convergence to 0. For the proof of the increasing of this norm we consider the tangent cone field $(|\dot{y}_s| > |\dot{y}_u|)$ at the graph of the function $\tilde{q}$. We use the fact that this cone field is $-(11)_{1,0}$- invariant in appropriate neighborhood of zero in this graph (and hence, so is the correspondent domain $K_{s,u} = \{|y_s| > |y_u|\} \cap \{y = \tilde{q}(t)\}$, whose closure contains the point $(\tilde{q}(t_0), t_0)$, by definition). The proof of these statements is analogous to that of proposition 3 in [6]. Now the proof of the $|\tilde{q}_s|$- increasing statement repeats that of the analogous statement from the proof of the asymptotic Taylor formula with the change of K_m to $K_{s,u}$. This contradicts the convergence to 0 of the trajectory from the beginning of the item and proves that $\tilde{q} \equiv 0$. Theorem 8 is proved.

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